A QUILT AFTER FIBONACCI, PART 3 OF 3: INTERSPERSOID–DISPERSOID ARRAYS AND GRAPHICAL COMPLEMENTARITY

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Dedicated to Patrícia

ABSTRACT. The *Fibonacci quilt* introduced in Part 1 of this paper is a geometric construction that serves as an abacus for certain integer sequences. This part of the paper studies the octet of two-dimensional arrays of integers that arise from the quilt, and a second octet of integer arrays identified in Part 2 of this paper. To study these arrays, the papers of Kimberling on interspersion–dispersion arrays (I–D arrays) and complementary equations serve as a guide.

While one of the quilt arrays turns out to be an I–D array, the other seven satisfy the relaxed definition of *interspersoid–dispersoid array*, given here. Results from Part 2 of this paper allow the development of row recurrences and dispersion parameters, completing the analogy between Kimberling's work on interspersions and dispersions, respectively, and their relaxed versions.

The paper also examines the block decomposition of the arrays induced by interspersion. For N greater than n, the rows N below row n break into blocks according the relative alignment of rows n and N in an interspersion of the two. For the quilt arrays, successive elements of row n themselves can be used to express the height of these blocks. This result exemplifies the self-similarity of these arrays and the quilt which generates them.

Although it shares one array with the first octet, the second octet comprises only I–D arrays. Moreover, some pairs of these arrays turn out to "mutually disperse" their first column(s). The mutual dispersion property coincides with duality between the arrays recognized in Part 2 of this paper.

Finally, the quilt provides a visualization of complementary equations, identities studied by Kimberling, and also discussed in Part 2.

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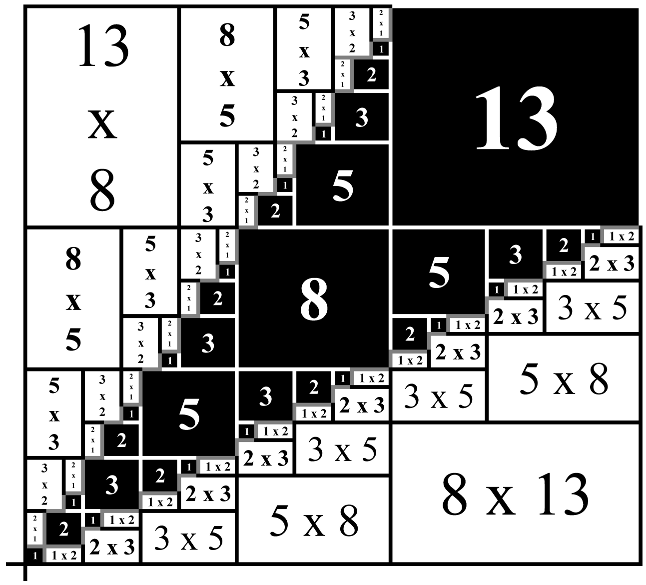


FIGURE 1. A Corner Patch of the Quilt Tiling

1. INTRODUCTION

This paper considers a quilt tiling after Fibonacci. The quilt comprises black squares and white rectangles of various sizes arranged in various positions with apparent self-similarity (Figure 1), providing a graphical calculus for recurrences within and between sequences. The identities visible in the quilt include complementary equations examined by Kimberling in [3] and [4], and revisited in Part 2 of this paper [6].

Superimposing the quilt on a unit-square grid provides coordinates for the border cells between its black and white regions — a complete characterization for the geometry of the border. In each row i, column indices $A(i) = 1, 2, 2, 3, 4, 4, 5, 5, 6, 7, \ldots,$ for the first black cell and $\Omega(i) = 1, 3, 4, 6, 8, 9, 11, 12, 14, \ldots,$ for the last black cell, respectively, describe all outside corners of the black region (Proposition 3.1).

Each outside corner of the black region lies in one and only one square of the quilt, and the set of squares partitions by size. Thus the quilt pattern partitions the set of black outside corner cells (Corollary 3.5). By means of this partition, the sequences of coordinates of these corner cells fall into two-dimensional arrays (Tables 1– 4). Analogously, the white rectangles partition outside corner cells of the white region into subsequences, complemented by the remaining corner cells of the white rectangles, thus yielding another quartet of two-dimensional arrays (Tables 5– 8).

Moreover, the sequence of horizontal coordinates of the black corner cells, is a spectrum sequence of the corresponding vertical coordinates, and vice versa (Corollary 3.4), giving the black region its characteristic shape. For the eight arrays of quilt coordinates, the paper will show that the first (Table 1) is an interspersion-dispersion array (I–D array) (Proposition 3.6) while the other seven (Tables 2– 8) are *interspersoids* and *dispersoids* (Definitions 3.1 and 3.2, respectively).

This part of the paper also revisits the Branch Quartet and Clade Quartet of I–D arrays (Tables 11, respectively, 12) identified in Part 2.

Part 1 of the paper [5] defined *cohort* as a group of squares or rectangles reproduced at an individual phase of constructing the quilt. In Part 2 of the paper [6], the grouping of squares and rectangles induced a cohort structure on the quilt subsequences, which provided a convenient calculation of the quilt arrays. Using cohort-based formulas (Proposition 3.2), this part of the paper derives row recurrences (Corollary 3.7) and dispersion parameters (Table 10), by analogy to the Kimberling's results on dispersions [1], [2], as well as row interspersion properties (Corollary 3.10 and 3.11), analogous to Kimberling's results on interspersions (*ibid*).

The paper also examines the block decomposition of arrays induced by interspersion, and reflected in the geometry of the quilt. For row n of an I–D array, rows $N > n$ below row n break into blocks according the relative alignment of rows n and N when the two are interspersed. That is, rows N and N' in the same block share the same relative position of their first element when interspersed with row n. For the quilt arrays, it turns out that successive elements of row n themselves give simple expressions for the *height* (number of rows) of successive blocks, and the starting and ending rows of each block. This result (Corollary 3.10) demonstrates yet one more element of self-similarity in the quilt and related sequences.

Dispersion of the first column plays a key role in Kimberling's work on I–D arrays [1], [2]. Part 2 of this paper identified two types of duality between arrays, *cohort duality* and *mirror duality* present in the Branch and Clade quartets. Here, it turns out that all mirror dual pairs (by definition) as well as certain cohort dual pairs mutually disperse each others' first column(s), Propositions 3.8 respectively 3.9, similar to how an I–D array self-disperses its own first column.

Part 2 showed the quilt's black squares to provide a graphical abacus for restricted compositions of integers (using only ones and twos). Considering columns of the quilt arrays as subsequences, this part of the paper also develops visual correspondences (Proposition 3.12) and visual complementarity (Proposition 3.13) of quilt subsequences. Being symmetric about its main diagonal, the quilt interchanges row and column coordinates, thus allowing visual alignment between any two of the quilt sequences.

Part 2 gave formulas to generate columns of the eight quilt arrays from sequences of compositions in the free monoid $\{\kappa, \lambda\}^\star$ generated by the Wythoff sequences $\kappa(n) = \lfloor n\phi \rfloor$ and $\lambda(n) = \lfloor n\phi^2 \rfloor$. The structure of this free monoid allowed the development of three canonical forms for cohort sequences. Here, these forms allow the visual identification within the quilt of complementary equations studied by Kimberling in [3] and [4]. Ultimately, the connection between quilt sequences and their cohort formulas also allows us to characterize their block decomposition (Corollary 4.6), and also underlies the spectrum property, Corollary 3.4.

2. Notation

General:

$\phi \equiv (\sqrt{5}+1)/2$,	The Golden Ratio;
$F_{k+1} = F_k + F_{k-1}$, $k \geq 1$, with $F_0 = 0$ and $F_1 = 1$	The Fibonacci numbers;
$L_{k+1} = L_k + L_{k-1}$, $k \geq 1$, with $L_0 = 2$ and $L_1 = 1$	The Lucas numbers;

123456,	Integer sequence A123456 from Sloane's OEIS [7];
$F^{-1}(n) \equiv$ 130233(n),	Greatest Fibonacci number $\leq n$;
The quilt (Figure 1):	
$(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$,	Coordinate pair for unit cell in row i and column j;
$A(i)$ such that cell $(i, A(i))$ is black and $(i, A(i)-1)$ is white;	Start of blackening in row i;
$\Omega(i)$ such that cell $(i, \Omega(i))$ is black and $(i, \Omega(i)+1)$ is white;	End of blackening in row i;
$[a,b] \times [c,d]$,	An interval of rows $\times$ columns;
$S_{0,k} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$	Black square $[a_{0,k}, b_{0,k}] \times [c_{0,k}, d_{0,k}]$ lying on the main diagonal;
$S_{n,k} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$,	A pair of equivalent squares: $[a_{n,k}, b_{n,k}] \times [c_{n,k}, d_{n,k}]$ below the diagonal, and $[c_{n,k}, d_{n,k}] \times [a_{n,k}, b_{n,k}]$ above the diagonal;
$R_{n,k} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$,	A pair of equivalent white quilt rectangles: $[\alpha_{1,k}, \beta_{1,k}] \times [\gamma_{1,k}, \delta_{1,k}]$ below the diagonal, and $[\gamma_{n,k}, \delta_{n,k}] \times [\alpha_{n,k}, \beta_{n,k}]$ above the diagonal;
$\boldsymbol{a} = (a_{n,k})_{n=0,k=1}^{\infty\ \ \infty} = (a_1 a_2 \cdots)$, $\boldsymbol{b} = (b_{n,k})_{n=0,k=1}^{\infty\ \ \infty} = (b_1 b_2 \cdots)$, $\boldsymbol{c} = (c_{n,k})_{n=0,k=1}^{\infty\ \ \infty} = (c_1 c_2 \cdots)$, $\boldsymbol{d} = (d_{n,k})_{n=0,k=1}^{\infty\ \ \infty} = (d_1 d_2 \cdots)$;	Quilt black coordinates as semi-infinite arrays, collection of scalar entries, and collection of columns;
$\boldsymbol{\alpha} = (\alpha_{n,k})_{n=1,k=1}^{\infty\ \ \infty} = (\alpha_1 \alpha_2 \cdots)$, $\boldsymbol{\beta} = (\beta_{n,k})_{n=1,k=1}^{\infty\ \ \infty} = (\beta_1 \beta_2 \cdots)$, $\boldsymbol{\gamma} = (\gamma_{n,k})_{n=1,k=1}^{\infty\ \ \infty} = (\gamma_1 \gamma_2 \cdots)$, $\boldsymbol{\delta} = (\delta_{n,k})_{n=1,k=1}^{\infty\ \ \infty} = (\delta_1 \delta_2 \cdots)$;	Quilt white coordinates, as semi-infinite arrays, collection of scalar entries, and collection of columns;
$a_k = (a_{n,k})_{n=0}^{\infty}$, $b_k = (b_{n,k})_{n=0}^{\infty}$, $c_k = (c_{n,k})_{n=0}^{\infty}$, $d_k = (d_{n,k})_{n=0}^{\infty}$;	k^{th} column of quilt black array,
$a_k^+ = (a_{n,k})_{n=1}^{\infty}$, $b_k^+ = (b_{n,k})_{n=1}^{\infty}$, $c_k^+ = (c_{n,k})_{n=1}^{\infty}$, $d_k^+ = (d_{n,k})_{n=1}^{\infty}$;	k^{th} column of quilt black array, zeroth element omitted;
$\alpha_k = (\alpha_{n,k})_{n=1}^{\infty}$, $\beta_k = (\beta_{n,k})_{n=1}^{\infty}$, $\gamma_k = (\gamma_{n,k})_{n=1}^{\infty}$, $\delta_k = (\delta_{n,k})_{n=1}^{\infty}$	k^{th} column of quilt white array;
Wythoff Sequences:	
$\kappa(n) = \lfloor n\phi \rfloor$, $\lambda(n) = \lfloor n\phi^2 \rfloor$,	Pair of Wythoff sequences;
$K = \kappa(\mathbb{Z}_+)$, $\Lambda = \lambda(\mathbb{Z}_+)$	Codomains of Wythoff compositions;

$K^2 = \kappa^2(\mathbb{Z}_+)$, $K\Lambda = \kappa\lambda(\mathbb{Z}_+)$, etc.	on $\mathbb{Z}_+$;
$S = (S_1, S_2, \ldots, S_n, \ldots)$	Cohort sequence with initial element S_1;
p	Growth rate parameter of a cohort sequence;

The branch and clade quartets (Tables 11 and 12):

F, ꟻ, Ⅎ, ᖵ	Branch quartet arrays;
w, 𝔴, a, ɒ	Clade quartet arrays;

0	0	1	2	4	7	12	20	33	54
1	1	3	5	9	15	25	41	67	109
2	3	6	10	17	28	46	75	122	198
3	4	8	13	22	36	59	96	156	253
4	6	11	18	30	49	80	130	211	342
5	8	14	23	38	62	101	164	266	431
6	9	16	26	43	70	114	185	300	486
7	11	19	31	51	83	135	219	355	575

TABLE 1. Table of $a_{n,k}$, for $n = 0, 1, \ldots, 7$ and $k = -1, 0, 1, 2 \ldots, 8$

−1	0	1	3	6	11	19	32	53	87
0	1	3	6	11	19	32	53	87	142
1	3	6	11	19	32	53	87	142	231
2	4	8	14	24	40	66	108	176	286
3	6	11	19	32	53	87	142	231	375
4	8	14	24	40	66	108	176	286	464
5	9	16	27	45	74	121	197	320	519
6	11	19	32	53	87	142	231	375	608

TABLE 2. Table of $b_{n,k}$, for $n = 0, 1, \ldots, 7$ and $k = -1, 0, 1, 2 \ldots, 8$

−1	0	0	1	2	4	7	12	20	33	54
0	1	2	4	7	12	20	33	54	88	143
1	3	5	9	15	25	41	67	109	177	287
2	4	7	12	20	33	54	88	143	232	376
3	6	10	17	28	46	75	122	198	321	520
4	8	13	22	36	59	96	156	253	410	664
5	9	15	25	41	67	109	177	287	465	753
6	11	18	30	49	80	130	211	342	554	897

TABLE 3. Table of $c_{n,k}$, for $n = 0, 1, \ldots, 7$ and $k = -2, -1, 0, 1, 2 \ldots, 8$

−1	−1	0	1	3	6	11	19	32	53	87
0	0	2	4	8	14	24	40	66	108	176
1	2	5	9	16	27	45	74	121	197	320
2	3	7	12	21	35	58	95	155	252	409
3	5	10	17	29	48	79	129	210	341	553
4	7	13	22	37	61	100	163	265	430	697
5	8	15	25	42	69	113	184	299	485	786
6	10	18	30	50	82	134	218	354	574	930

TABLE 4. Table of $d_{n,k}$, for $n = 0, 1, \ldots, 7$ and $k = -2, -1, 0, 1, 2 \ldots, 8$

1	1	1	1	1	1	1	1	1
2	3	4	6	9	14	22	35	56
3	4	6	9	14	22	35	56	90
4	6	9	14	22	35	56	90	145
5	8	12	19	30	48	77	124	200
6	9	14	22	35	56	90	145	234
7	11	17	27	43	69	111	179	289
8	12	19	30	48	77	124	200	323

TABLE 5. Table of $\alpha_{n,k}$, for $n = 1, \ldots, 8$ and $k = 0, 1, \ldots, 8$

1	1	2	3	5	8	13	21	34
2	3	5	8	13	21	34	55	89
3	4	7	11	18	29	47	176	123
4	6	10	16	26	42	68	110	178
5	8	13	21	34	55	89	144	233
6	9	15	24	39	63	102	165	267
7	11	18	29	47	76	123	199	322
8	12	20	32	52	84	136	220	356

TABLE 6. Table of $\beta_{n,k}$, for $n = 1, \ldots, 8$ and $k = 0, 1, \ldots, 8$

3. MAIN RESULTS

3.1. **Quilt results.** The following two propositions appear in Part 1 [5], respectively, Part 2 [6] of this paper:

Proposition 3.1. *Consider the grid of unit square cells underlying Figure 1. For $n = 1, 2, \ldots$, blackening in row (column) n of the Figure begins in column (row) $A(n) = \lceil n/\phi \rceil$ and ends in column (row) $\Omega(n) = \lfloor n\phi \rfloor \equiv \kappa(n)$.*

Proof. Proved in Part 1 of this paper [5]. □

0	1	2	4	7	12	20	33	54	88
1	3	5	9	15	25	41	67	109	177
2	4	7	12	20	33	54	88	143	232
3	6	10	17	28	46	75	122	198	321
4	8	13	22	36	59	96	156	253	410
5	9	15	25	41	67	109	177	287	465
6	11	18	30	49	80	130	211	342	554
7	12	20	33	54	88	143	232	376	609

TABLE 7. Table of $\gamma_{n,k}$, for $n = 1, \ldots, 8$ and $k = -1, 0, 1 \ldots, 8$

0	1	3	6	11	19	32	53	87	142
1	3	6	11	19	32	53	87	142	231
2	4	8	14	24	40	66	108	176	286
3	6	11	19	32	53	87	142	231	375
4	8	14	24	40	66	108	176	286	464
5	9	16	27	45	74	121	197	320	519
6	11	19	32	53	87	142	231	375	608
7	12	21	35	58	95	155	252	409	663

TABLE 8. Table of $\delta_{n,k}$, for $n = 1, \ldots, 8$ and $k = -1, 0, 1, \ldots, 8$

Proposition 3.2 (Cohort-based formulas). *For the black squares in Figure 1, $n = 0, 1, 2, \ldots$, $k = 1, 2, 3 \ldots$,*

$$a_{n,k} = F_{k+2} + \kappa(n) F_{k+1} + n F_k \quad -1 \stackrel{n \geq 0}{=\!=} \kappa^{k+1}(n) + 2F_{k+2} - 2; \tag{1}$$

$$b_{n,k} = F_{k+3} + \kappa(n) F_{k+1} + n F_k \quad -2 \stackrel{n \geq 0}{=\!=} \kappa^{k+1}(n) \; + F_{k+4} - 3; \tag{2}$$

$$c_{n,k} = F_{k+2} + \kappa(n) F_{k+2} + n F_{k+1} - 1 \stackrel{n \geq 0}{=\!=} \kappa^{k+2}(n) \; + F_{k+4} - 2; \tag{3}$$

$$d_{n,k} = F_{k+3} + \kappa(n) F_{k+2} + n F_{k+1} - 2 \stackrel{n \geq 0}{=\!=} \kappa^{k+2}(n) + 2F_{k+3} - 3;. \tag{4}$$

Whereas, for the white rectangles in Figure 1, $n = 1, 2, 3 \ldots$, $k = 1, 2, 3 \ldots$,

$$\alpha_{n,k} = -F_{k+1} + \kappa(n) F_k \quad + n F_{k-1} + 1 = \kappa^k(n); \tag{5}$$

$$\beta_{n,k} = \qquad \kappa(n) F_k \quad + n F_{k-1} \quad = \kappa^k(n) \quad + F_{k+1} - 1; \tag{6}$$

$$\gamma_{n,k} = \quad F_{k+1} + \kappa(n) F_{k+1} + n F_k \quad -1 = \kappa^{k+1}(n) + F_{k+3} - 2; \tag{7}$$

$$\delta_{n,k} = \quad F_{k+3} + \kappa(n) F_{k+1} + n F_k \quad -2 = \kappa^{k+1}(n) + F_{k+4} - 3. \tag{8}$$

For related sequences, $n = 0, 1, 2, \ldots$, $k = 1, 2, 3 \ldots$,

$$w_{n-1,k} = - \quad F_k + \kappa(n) F_{k+1} \; + n F_k \qquad \stackrel{n \geq 0}{=\!=} \kappa^{k+1}(n) \quad + F_{k+1} - 1;$$

$$\hat{\mathfrak{w}}_{n-1,k} = - F_{2k-2} + \kappa(n) F_{2k-1} + n F_{2k-2} - 1 \stackrel{n \geq 0}{=\!=} \kappa^{2k-1}(n) + F_{2k-1} - 2.$$

$$\mathfrak{v}_{n,k} = \qquad F_{2k} + \kappa(n) F_{2k-1} + n F_{2k-2} \qquad \stackrel{n \geq 0}{=\!=} \kappa^{2k-1}(n) \; + 2F_{2k} - 1.$$

In each case, the first expression gives the cohort form, *while the second gives the* pure-κ form.

Proof. In Part 2 of this paper [6]. □

Corollary 3.3 (Cross-lags between $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$)**.** $k = 1, 2, 3, \ldots,$

$$\begin{aligned}
a_{n,k} &= a_{n,k-1} + a_{n,k-2} + 1 = b_{n,k} - F_{k+1} + 1 &&= c_{n,k-1} + F_k &&= d_{n,k-1} + 1 \\
b_{n,k} &= a_{n,k} + F_{k+1} - 1 \quad = b_{n,k-1} + b_{n,k-2} + 2 &&= c_{n,k-1} + F_{k+2} - 1 &&= d_{n,k-1} + F_{k+1} \\
c_{n,k} &= a_{n,k+1} - F_{k+1} \quad = b_{n,k+1} - F_{k+3} + 1 &&= c_{n,k-1} + c_{n,k-2} + 1 &&= d_{n,k} - F_{k+1} + 1 \\
d_{n,k} &= a_{n,k+1} - 1 \quad = b_{n,k+1} - F_{k+2} &&= c_{n,k} + F_{k+1} - 1 &&= d_{n,k-1} + d_{n,k-2} + 2
\end{aligned}$$

Corollary 3.4 (Spectrum relationship between $\boldsymbol{a}$ and $\boldsymbol{d}$)**.**

$$d_{n,k} = \lfloor a_{n,k}\phi \rfloor \equiv \kappa(a_{n,k}), n = 0, 1, 2, \ldots, k = 1, 2, 3 \ldots. \tag{9}$$

Proof. Given in Section 4 following Proposition 4.4. □

Corollary 3.5. *For Figure 1, consider the underlying grid of unit cells. Now for each row i of the grid, consider the rightmost black unit cell in the row. The coordinates (i, Ω_i) for the black unit cells can be written as the union of the coordinates $(a_{n,k}, d_{n,k})$ for the southeastern corners of the black squares $S_{n,k}$ on or below the diagonal in Figure 1, or equivalently, the northwestern corners of all black squares on or above the diagonal. That is, $\{(i, \Omega_i)\}_{i=1}^{\infty} = \{(a_{n,k}, d_{n,k})\}_{n=0,k=1}^{\infty,\ \infty}$.*

Proof. Follows from Corollary 3.4 and Proposition 3.6. □

3.2. **Interspersion & dispersion properties of the eight quilt arrays.**

Proposition 3.6 ($\boldsymbol{a}$ is an Interspersion–Dispersion Array)**.**

Proof. Let $T(n, k)$ refer to 083047 in Sloane [7]. By Proposition 3.2, $a_{n,1} = \lfloor n\phi \rfloor + n + 1 = \lfloor n(\phi + 1) \rfloor + 1 = T(n, 0)$. By Corollary 3.3, $\boldsymbol{a}$ is a lagged version of $\boldsymbol{d}$, while by Corollary 3.4, $\boldsymbol{d}$ is also a spectrum sequence in $\boldsymbol{a}$. Thus, $a_{n,k+1} = d_{n,k} + 1 = \lfloor a_{n,k}\phi \rfloor + 1 = \lceil a_{n,k}\phi \rceil$ showing that $a_{n,k} = T_{n,k-1}$, as defined in [7]. Whereas $(T(n, k))_{n=0,k=0}^{\infty,\ \infty}$ is an interspersion, so is $\boldsymbol{a} = (a_{n,k})_{n=0,k=1}^{\infty,\ \infty}$. □

Remark 3.1. Since $\boldsymbol{a}$ is an interspersion, it is also a dispersion (see Kimberling [1]). From (1), obtain $a_{n,1} = \kappa(n) + n + 1$, $n \geq 0$, giving the first column $a_1 =$ (1, 3, 6, 8, 11, 14, 16, 19,…) of $\boldsymbol{a}$. The ordered complement of a_1 in the positive integers, $s_n =$ 2, 4, 5, 7, 9, 10, 12, 13,…, is given by $s_n = \kappa(n) + 1$ and is dispersed among the remaining columns via $a_{n,k} = s_{a_{n,k-1}}$, $k \geq 1, n \geq 0$.

3.2.1. *Kimberling's Interspersion–Dispersion Properties.* Regarding Kimberling's four interspersion properties [1], the arrays $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ (Tables 2, 3, and 4) satisfy the last three: Properties (I2), (I3) and (I4). In particular, arrays $\boldsymbol{b}$ and $\boldsymbol{c}$ satisfy Kimberling's (I4) in as much as the interspersion of a term of one row between two consecutive terms of another implies that the interspersion of the two rows continues for the remainder of the rows (Property (I4a) of Definition 3.1, here). For arrays $\boldsymbol{b}$ and $\boldsymbol{c}$, moreover, distinct rows may also coincide rather than intersperse (see Corollaries 3.10 and 3.11). Where coincidence occurs, rows of $\boldsymbol{b}$ and $\boldsymbol{c}$ satisfy a generalization of Kimberling's (I4), in that if two terms from distinct rows coincide, then the coincidence continues for the remainder of the rows (Property (I4b) of Definition 3.1).

However, none of the arrays $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ contains all of the positive integers (Kimberling's first interspersion property (I1)), and $\boldsymbol{b}$ and $\boldsymbol{c}$ in particular contain multiple instances of the same entries occurring in different rows.

Furthermore, regarding Kimberling's four dispersion properties, each of the arrays $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ satisfies the first, Property (D1), in that the first column of each strictly increases. As well, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ satisfy the second, Property (D2), in that the second element of the first row is ≥ 2. However, array $\boldsymbol{b}$ fails to satisfy Kimberling's (D3) and (D4), in that it does not disperse the complement of its first column. Rather, the first column b_1 of $\boldsymbol{b}$ contains all entries of subsequent columns (Corollary 3.11). Proposition 3.13 (30) gives a visualization of this (Figure 15).

For array $\boldsymbol{c}$, the second column is a complement of the first, although their union is not the positive integers, $c_1 \cup c_2 \neq \mathbb{Z}_{\geq 1}$. Rather $c_1 \cup c_2 = \{1{,}2\} \cup [\Lambda + 2]$ (see Figure 9). Moreover, the first two columns of $\boldsymbol{c}$ together contain all entries of subsequent columns (Corollary 3.11). Proposition 3.13 (29) gives a visualization of this (Figure 14). Thus, $\boldsymbol{c}$ does "disperse" the complement of its first column, though in a manner more general than Kimberling's Property (D3) contemplates.

All four quilt black arrays ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$) satisfy a more general version of Kimberling's (D4) in the generic sense that there exists a dispersion structure or recurrence in the first column for the remaining columns, stated in Definition 3.2, and with parameters given in Table 10. In particular, for array $\boldsymbol{a}$ this reduces to Kimberling's original (D4) per Remark 3.1, while array $\boldsymbol{d}$ can be said to disperse $s_n = \lambda(n) + 1$, $n \geq 1$, which has empty intersection with the first column $d_1 = 2\kappa(n) + n + 1$ of $\boldsymbol{d}$, among its remaining columns. In this case, s_n is not the complement of the first column d_1 in the positive integers $\mathbb{Z}_{\geq 1}$, rather its complement in K. Proposition 3.13 (27), (30) produces a visualization of this (Figure 15).

		S Arrays				R Arrays			
		$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$
Interspersion Properties	I1	✓							
	I2	✓	✓	✓	✓		✓	✓	✓
	I3	✓	✓	✓	✓	✓	✓	✓	✓
	I4	✓	✓	✓	✓	✓	✓	✓	✓
Dispersion Properties	D1	✓	✓	✓	✓	✓	✓	✓	✓
	D2	✓	✓	✓	✓		✓	✓	✓
	D3	✓		✓	✓		✓	✓	
	D4	✓			✓				

TABLE 9. Kimberling's Interspersion–Dispersion properties [1] for the coordinate arrays of black squares S (Tables 1–4) and white rectangles R (Tables 5–8) in the quilt (Figure 1)

None of the arrays $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ contains all of the positive integers. Though the array $\boldsymbol{\beta}$ does contain all the positive integers, its rows are not a partition. Hence, none of the quilt white arrays satisfies Kimberling's (I1). The first row of $\boldsymbol{\alpha}$ contains only ones, whilst all other rows and columns of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ strictly increase. Thus, $\boldsymbol{\alpha}$ satifies Kimberling's (I3), whilst $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ satisfy both (I2) and (I3). The four quilt white arrays, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, also satisfy a version of Kimberling's (I4), with the same provision for coincidence of rows that Property (I4b) made for $\boldsymbol{b}$ and $\boldsymbol{c}$.

For $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, the second entry in the first row is ≥ 2 (Kimberling's (D2)). Arrays $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, resemble $\boldsymbol{b}$, in that the first column contains all entries of subsequent columns, whilst $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ resemble $\boldsymbol{c}$, in as much as the first two columns do not share entries, but together contain all elements of subsequent columns (Corollary 3.11). For $\boldsymbol{b}$, $\beta_1 \cup \beta_2 = \mathbb{Z}_{\geq 1}$ (Figure 12), while Proposition 3.13 (28) produces a visualization of $\gamma_1 \cup \gamma_2 = K+1$ (Figure 13). Thus, $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ fail to satisfy Kimberling's (D3), whilst $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ do "disperse" the complement of their first columns, albeit in a more general way than Kimberling's (D3) implies.

Table 9 summarizes the interspersion and dispersion properties for the eight quilt arrays. In contrast to $\boldsymbol{a}$ (Proposition 3.6), none of the other quilt arrays, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ or $\boldsymbol{\delta}$, is an I–D array as defined by Kimberling, yet each has a distinct structure with respect to the properties and identities discussed in [1] and [2]. This motivates the relaxed definitions of *interspersoid* and *dispersoid*, Definitions 3.1 and 3.2, next.

3.2.2. *Relaxed Definitions of Interspersoid and Dispersoid.*

Definition 3.1 (Interspersoid)**.** By analogy to the interspersion that Kimberling introduced and defined by four properties [1], define an array $(e_{i,j})$ of integers to be an *interspersoid* if

- I1. $(e_{i,j})$ is a subset of the positive integers;
- I2. Every row of $(e_{i,j})$ is a strictly increasing sequence;
- I3. Every column of $(e_{i,j})$ is a strictly increasing sequence;
- I4a. If n and N are indices of distinct rows of $(e_{i,j})$, and if k and h are any indices for which $e_{n,k} < e_{N,h} < e_{n,k+1}$, then $e_{n,k+1} < e_{N,h+1} < e_{n,k+2}$.
- I4b. If n and N are indices of distinct rows of $(e_{i,j})$, and if k and h are any indices for which $e_{n,k} = e_{N,h}$, then $e_{n,k+1} = e_{N,h+1}$.

Definition 3.2 (Dispersoid)**.** By analogy to the dispersion that Kimberling introduced and defined by four properties [1], define an array $(e_{i,j})$, $j \geq 1$ of integers to be a *dispersoid* if

- D1. The first column of $(e_{i,j})$ is a strictly increasing sequence;
- D2. Relaxed;
- D3. Relaxed;
- D4. $e_{i,j} = s(e_{i,j-1})$, for all $j \geq 2$, where $s()$ is an integer valued function that is defined and strictly increasing on a set of integers that is closed under s and that contains the first column of $(e_{i,j})$.

		$s(n)$		
		$\lfloor n\phi \rfloor$	$\lfloor n\phi \rfloor + 1$	$\lfloor n\phi \rfloor + 2$
First Column	$\lfloor n\phi \rfloor$	$\boldsymbol{\alpha}$,$\boldsymbol{\beta}^{\dagger}$	$\boldsymbol{\beta}^{\dagger}$	
	$\lfloor n\phi \rfloor + n$		$\boldsymbol{\gamma}$	
	$\lfloor n\phi \rfloor + n + 1$		$\boldsymbol{a}$	$\boldsymbol{b}, \boldsymbol{\delta}$
	$2\lfloor n\phi \rfloor + n + 1$		$\boldsymbol{c}$	$\boldsymbol{d}$

†The form of $s(n)$ for $\boldsymbol{\beta}$ alternates columns.

Table 10. Dispersoid parameters (Definition 3.2) for arrays $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$, obtained from Proposition 3.2 and Corollary 3.7

For the eight quilt arrays, Table 10 summarizes the dispersoid parameters corresponding to Definition 3.2.

Remark 3.2. Comparing (2) and (8), observe that the array $\boldsymbol{\delta} = (b_{n,k})_{n,k=1}^{\infty}$ is the same as the array $\boldsymbol{b}$ with the 0^{th} row removed, that is, $\boldsymbol{\delta} = (b_1^+ b_2^+ \cdots)$. For economy, therefore, the following discussion does not explicitly treat $\boldsymbol{\delta}$. Similarly, comparing (3) and (7), observe that the array $\boldsymbol{\gamma} = (c_{n,k-1})_{n,k=1}^{\infty}$ is the same as the array $\boldsymbol{c}$ with the 0^{th} row removed and a 0^{th} column prepended, that is, $\boldsymbol{\gamma} = (c_0^+ c_1^+ \cdots)$.

3.3. **The branch and clade quartets.** Tables 11 and 12 present the branch and clade quartets of I–D arrays, comprising F, ꟻ, Ⅎ, ⅃, and w, ɯ, a, ɒ, respectively. Part 2 of the paper [6] classified these arrays based on pairwise "clade duality" and "mirror duality." and Section 3.5 here examines the mutual-dispersion property of dual pairs of the arrays.

⟸ Cohort Duality ⟹

⟸ Mirror Duality ⟹

1	2	3	5	8	13	1	2	4	7	12	20
4	6	9	14	22	35	3	6	11	19	32	53
7	10	15	23	36	57	5	10	18	31	52	86
11	16	24	37	58	92	8	16	29	50	84	139
12	17	25	38	59	93	9	17	30	51	85	140
18	26	39	60	94	149	13	26	47	81	136	225
19	27	40	61	95	150	14	27	48	82	137	226
20	28	41	62	96	151	15	28	49	83	138	227
29	42	63	97	152	241	21	42	76	131	220	364
30	43	64	98	153	242	22	43	77	132	221	365
1–2-Fibonacci Array(F)						**2–1-Fibonacci Array**(Ⅎ)					
$n + F_{F^{-1}(n)+k+1}$ (194030)						$n + F_{F^{-1}(n)+k+2} - F_{F^{-1}(n)+2}$					
1	4	12	33	88	232	1	3	8	21	55	144
2	7	20	54	143	376	2	5	13	34	89	233
3	11	32	87	231	608	4	9	22	56	145	378
5	18	52	141	374	984	6	14	35	90	234	611
6	19	53	142	375	985	7	15	36	91	235	612
8	29	84	228	605	1592	10	23	57	146	379	989
9	30	85	229	606	1593	11	24	58	147	380	990
10	31	86	230	607	1594	12	25	59	148	381	991
13	47	136	369	979	2576	16	37	92	236	613	1600
14	48	137	370	980	2577	17	38	93	237	614	1601
1–2 Mirror Array(ꟻ)						**2–1 Mirror Array**(⅃)					
$F_{2k+1} - 1,\ n=0$; $n+F_{F^{-1}(n)+2k} - 2F_{F^{-1}(n)},\ n \geq 1$.						$F_{2k},\ n=0$; $n+F_{F^{-1}(n)+2k-1} - F_{F^{-1}(n)-1},\ n \geq 1$.					

TABLE 11. Branch Quartet of I–D arrays [6]

3.4. **Results on dispersoids.**

Corollary 3.7 (of Proposition 4.2. Recurrences within Rows)**.**

$\Longleftarrow$ Cohort Duality $\Longrightarrow$

$\Longleftarrow$ Mirror Duality $\Longrightarrow$

1	2	3	5	8	13	1	2	4	7	12	20
4	7	11	18	29	47	3	5	9	15	25	41
6	10	16	26	42	68	6	10	17	28	46	75
9	15	24	39	63	102	8	13	22	36	59	96
12	20	32	52	84	136	11	18	30	49	80	130
14	23	37	60	97	157	14	23	38	62	101	164
17	28	45	73	118	191	16	26	43	70	114	185
19	31	50	81	131	212	19	31	51	83	135	219
22	36	58	94	152	246	21	34	56	91	148	240
25	41	66	107	173	280	24	39	64	104	169	274
Wythoff Array, (w) (**035513**)						**Quilt Array,** (a) (**083047**)					
$F_{k+1}\kappa(n+1)+F_k n, n \geq 0$						$F_{k+1}\kappa(n)+F_k n+F_{k+2}-1, n \geq 0$					
1	4	12	33	88	232	1	3	8	21	55	144
2	6	17	46	122	321	2	6	16	42	110	288
3	9	25	67	177	465	4	11	29	76	199	521
5	14	38	101	266	698	5	14	37	97	254	665
7	19	51	135	355	931	7	19	50	131	343	898
8	22	59	156	410	1075	9	24	63	165	432	1131
10	27	72	190	499	1308	10	27	71	186	487	1275
11	30	80	211	554	1452	12	32	84	220	576	1508
13	35	93	245	643	1685	13	35	92	241	631	1652
15	40	106	279	732	1918	15	40	105	275	720	1885
Wythoff Mirror Array, $(\mathfrak{w})$						**Quilt Mirror Array,** $(\mathfrak{a})$					
$F_{2k+1}-1,\ n=0;$ (191436)						(132827)					
$F_{2k-1}\kappa(n+1)+F_{2k-2}n-1, n \geq 1.$						$F_{2k-1}\kappa(n)+F_{2k-2}n+F_{2k}, n \geq 0.$					

TABLE 12. Clade Quartet of I–D arrays [6]

(1) *By analogy to Kimberling's results on dispersions [1], the following row recurrences hold for* $n = 0, 1, 2, \ldots, k = 1, 2, 3, \ldots,$*:*

$$
\begin{aligned}
a_{n,k+1} &= \kappa(a_{n,k})+1, \quad a_{n,k+2} = \lambda(a_{n,k})+2;\\
b_{n,k+1} &= \kappa(b_{n,k})+2, \quad b_{n,k+2} = \lambda(b_{n,k})+4, \quad b_{n,k+3} = \lfloor b_{n,k}\phi^3 \rfloor + 7;\\
c_{n,k+1} &= \kappa(c_{n,k})+1, \quad c_{n,k+2} = \lambda(c_{n,k})+2;\\
d_{n,k+1} &= \kappa(d_{n,k})+2, \quad d_{n,k+2} = \lambda(d_{n,k})+4;
\end{aligned}
$$

while for $n = 1, 2, 3, \ldots, k = 1, 2, 3, \ldots$*:*

$$
\begin{aligned}
\alpha_{n,k+1} &= \kappa(\alpha_{n,k}), \qquad \alpha_{n,k+2} = \lambda(\alpha_{n,k})-1;\\
\beta_{n,k+1} &= \begin{cases} \kappa(\beta_{n,k})+1, \\ \kappa(\beta_{n,k}), \end{cases} \quad \beta_{n,k+2} = \begin{cases} \lambda(\beta_{n,k})+1, & k \text{ odd;} \\ \lambda(\beta_{n,k}), & k \text{ even;} \end{cases}\\
\gamma_{n,k+1} &= \kappa(\gamma_{n,k})+1, \quad \gamma_{n,k+2} = \lambda(\gamma_{n,k})+2;
\end{aligned}
$$

and for $n = 0, 1, 2, \ldots$*:*

$$w_{n,k+h} = \begin{cases} \lfloor w_{n,k}\phi^h \rfloor + 1, & k \geq h-1 \textit{ odd;} \\ \lfloor w_{n,k}\phi^h \rceil, & k \geq h-1 \textit{ even;} \end{cases}$$
$$\text{ɒ}_{n,k+1} = \lambda(\text{ɒ}_{n,k}) + 1, \quad k \geq 1;$$
$$\text{ʍ}_{n,k+1} = \lambda(\text{ʍ}_{n,k}) + 2, \quad k \geq 2.$$

(2) *Row recursions by analogy to Lemma 2.3 of Kimberling [2], are:*

$$\begin{array}{lll} a_{n,k} = & \lfloor a_{n,k+h}/\phi^h \rceil, & h=1,2,3; \\ b_{n,k} = \lfloor b_{n,k+1}/\phi \rceil = & \lceil b_{n,k+h}/\phi^h \rceil - 1, & h=2,3,4,5,6,7; \\ c_{n,k} = & \lceil c_{n,k+h}/\phi^h \rfloor, & h=1,2,3,4,5; \\ d_{n,k} = \lfloor d_{n,k+1}/\phi \rceil = & \lceil d_{n,k+h}/\phi^h \rceil - 1, & h=2,3,4,5. \end{array}$$

$$\begin{array}{llll} \alpha_{n,k} = & \lfloor \alpha_{n,k+h}/\phi^h \rfloor + 1, & & h=1,2,3; \\ \beta_{n,k} = & \begin{cases} \lfloor \beta_{n,k+h}/\phi^h \rceil, & k \textit{ odd;} \\ \lceil \beta_{n,k+h}/\phi^h \rfloor + 1, & k \textit{ even;} \end{cases} & & \forall h \geq 1; \\ \gamma_{n,k} = & \lceil \gamma_{n,k+h}/\phi^h \rceil, & & h=1,2,3. \end{array}$$

$$\begin{array}{lll} w_{n,k} = & \begin{cases} \lfloor w_{n,k+h}/\phi^h \rfloor, & k \geq h-1 \textit{ odd;} \\ \lceil w_{n,k+h}/\phi^h \rfloor + 1, & k \geq h-1 \textit{ even;} \end{cases} & \forall h \geq 1; \end{array}$$
$$\text{ɒ}_{n,k} = \lfloor \text{ɒ}_{n,k+1}/\phi^2 \rfloor;$$
$$\text{ʍ}_{n,k} = \lfloor \text{ʍ}_{n,k+1}/\phi^2 \rceil.$$

Proof.

(1) Analyze Proposition 4.2 and note the initial values of h for which no integers lie between the upper and lower bounds.

(2) Analyze Corollary 4.3 and note the initial values of h for which no integers lie between the upper and lower bounds.

□

3.5. **Results on mutual dispersion.** To avoid confusion with several other types of duality, [6] adopted the term "mirror duality" for what Kimberling [1] called the "inverse I–D array" of an I–D array with infinitely many rows. Once established that two arrays are I–D arrays and are mirror duals of one another, then as an immediate consequence, the two arrays mutually disperse one another's first columns.

Mutual dispersion between two arrays can also be proven directly, as the paper does below, without first demonstrating that the arrays satisfy the I–D properties. Writing out this mutual dispersion using indices of the array entries, the present work notes that an analogous mutual dispersion holds between pairs of certain "cohort dual" arrays.

3.5.1. *Mutual dispersion property of mirror duals.* Part 2 [6] observed that for pairs of "mirror duals" in the Branch Quartet (Table 11):

$$\mathsf{F}_1 \cup \text{ꟻ}_1 = \mathbb{Z}_+,\ \mathsf{F}_1 \cap \text{ꟻ}_1 = \{1\}; \tag{10}$$
$$\mathsf{L}_1 \cup \text{⅃}_1 = \mathbb{Z}_+,\ \mathsf{L}_1 \cap \text{⅃}_1 = \{1\};$$

and that for pairs of "mirror duals" in the Clade Quartet (Table 12):

$$w_1 \cup \text{ʍ}_1 = \mathbb{Z}_+,\ w_1 \cap \text{ʍ}_1 = \{1\}; \tag{11}$$
$$a_1 \cup \text{ɒ}_1 = \mathbb{Z}_+,\ a_1 \cap \text{ɒ}_1 = \{1\}.$$

Thus for any of the eight arrays in the Branch and Clade Quartets, all entries of a column $k \geq 2$ are found within the first column of its mirror dual. Moreover, its own previous column indexes the entries in the first column of its mirror dual, and reciprocally, giving the property stated in Proposition 3.8.

While the mutual dispersion property holds between any I–D array with infinitely many rows and its inverse, the pairs of mirror-dual arrays of the branch and clade quartets are singled out here, because an analogous property holds between cohort-dual arrays in the quartets, addressed in the next section, and because the proofs of both results resemble those of other results given here.

Proposition 3.8 (Mutual dispersion property of mirror duals in the quartets)**.**

$$
\begin{aligned}
\mathsf{F}_{n,k} &= \reflectbox{F}_{\mathsf{F}_{n,k-1},1},\ k \geq 2;\ \textit{and}\ \reflectbox{F}_{n,k} = \mathsf{F}_{\reflectbox{F}_{n,k-1},1},\ k \geq 2;\\
\scalebox{1}[-1]{F}_{n,k} &= \rotatebox[origin=c]{180}{F}_{\scalebox{1}[-1]{F}_{n,k-1},1},\ k \geq 2;\ \textit{and}\ \rotatebox[origin=c]{180}{F}_{n,k} = \scalebox{1}[-1]{F}_{\rotatebox[origin=c]{180}{F}_{n,k-1},1},\ k \geq 2;\\
w_{n,k} &= \reflectbox{w}_{w_{n,k-1},1}, k \geq 2;\ \textit{and}\ \reflectbox{w}_{n,k} = w_{\reflectbox{w}_{n,k-1},1}, k \geq 2;\\
a_{n,k} &= \reflectbox{a}_{a_{n,k-1},1},\ k \geq 2;\ \textit{and}\ \reflectbox{a}_{n,k} = a_{\reflectbox{a}_{n,k-1},1},\ k \geq 2.
\end{aligned}
$$

Proof. Part 2 of this paper [6] showed this result using a definition of the array rows as sequences of all-left or all-right branchings in binary trees, which implicitly guarantees Kimberling's dispersion property (D4). A different proof appears here in Section 4 using the formulas for the array entries given in Tables 11 and 12 and a general result (Lemma 4.5) about the Fibonacci cohort of array entries. The steps parallel those of proofs of other results in this third part of the paper. □

Proposition 3.8 can also be described by analogy to the ID–properties of arrays developed by Kimberling [1].

For instance, the first column of $\reflectbox{$\mathsf{F}$}$ plays the role of Kimberling's s for F, in that F disperses the first column of $\reflectbox{$\mathsf{F}$}$ — property (D4) of [1] — and vice versa. This hinges on the observation (10) that the first columns of F and $\reflectbox{$\mathsf{F}$}$ are (almost) complements of one another in $\mathbb{Z}_+$, similar to dispersion property (D3), and that $\mathsf{F}_{0,2} = \reflectbox{F}_{1,1} = 2 \geq 2$ and conversely, $\reflectbox{$\mathsf{F}$}_{0,2} = \mathsf{F}_{1,1} = 4 \geq 2$, similar to dispersion property (D2).

Likewise, the first column of $\reflectbox{$w$}$ plays the role of s in (D4) for w, whereas w disperses the first column of $\reflectbox{$w$}$, and vice versa. Here again, the similarity to dispersion properties continues. Similar to (D3), the first columns of w and $\reflectbox{$w$}$ are almost complements of one another in $\mathbb{Z}_+$ (11). Similar to (D2), $w_{0,2} = \reflectbox{$w$}_{1,1} = 2 \geq 2$ and $\reflectbox{$w$}_{0,2} = w_{1,1} = 4 \geq 2$.

3.5.2. *Mutual dispersion property of cohort duals.* Another mutual dispersion property emerges between pairs of cohort-dual arrays in the branch quartet and clade quartets. For w and a, the property alternates columns, whilst for F and $\scalebox{1}[-1]{$\mathsf{F}$}$, the property takes a form similar to how mirror duals disperse one another's first columns and how an I–D array self-disperses the complement of its own first column.

Proposition 3.9 (Mutual dispersion property of certain cohort duals in the quartets)**.**

$$
\mathsf{F}_{n,k} = \scalebox{1}[-1]{F}_{\mathsf{F}_{n,k-2},1}, \quad k \geq 3; \qquad \textit{and}\ \scalebox{1}[-1]{F}_{n,k} = \mathsf{F}_{\scalebox{1}[-1]{F}_{n,k-2},1}, \quad k \geq 3; \tag{12}
$$

$$
w_{n,k} = \begin{cases} a_{w_{n,k-2},1}, & k \geq 3,\ \textit{odd};\\ a_{w_{n,k-3},2}, & k \geq 4,\ \textit{even}; \end{cases} \quad \textit{and}\ a_{n,k} = \begin{cases} w_{a_{n,k-2},1}, & k \geq 3,\ \textit{odd};\\ w_{a_{n,k-3},2}, & k \geq 4,\ \textit{even}; \end{cases} \tag{13}
$$

Proof. Part 2 of this paper [6] sketched a proof using tree branches. A different proof appears in Section 4. □

Starting with the pair F and $\scalebox{1}[-1]{$\mathsf{F}$}$,

$$\begin{aligned}\mathsf{F}_1 \cup \mathsf{F}_2 \cup \scalebox{1}[-1]{F}_1 \ \cup \scalebox{1}[-1]{F}_2 &= \mathbb{Z}_+,\\ \mathsf{F}_1 \cap \scalebox{1}[-1]{F}_1 \qquad &= \{1\},\\ \mathsf{F}_2 = \scalebox{1}[-1]{F}_2. &\end{aligned}$$

Thus, in either of the arrays F and $\scalebox{1}[-1]{$\mathsf{F}$}$, all entries of a column $k \geq 2$ are found within the first *two* columns of its cohort dual, analogous to a relaxation of (D3). Specifically for column $k = 2$, its entries are identical to those in column two of the cohort dual. Analogous to (D4), for a column $k \geq 3$, its own *second-previous* column indexes its entries in the first column of the mirror dual, and reciprocally, leading to (12).

For $\reflectbox{$\mathsf{F}$}$ and $\rotatebox[origin=c]{180}{F}$, the second columns have empty intersection, while the first columns collectively contain all positive integers:

$$\begin{aligned}\reflectbox{F}_1 \cup \rotatebox[origin=c]{180}{F}_1 &= \mathbb{Z}_+,\\ \reflectbox{F}_2 \cap \rotatebox[origin=c]{180}{F}_2 &= \emptyset.\end{aligned}$$

Thus, for a column $k \geq 2$ of $\reflectbox{$\mathsf{F}$}$ or $\rotatebox[origin=c]{180}{F}$, all of its entries are found in the first column of the other array, a version of dispersion property (D3). For the mutual dispersion, an analogy to property (D4) is tenuous, as the dispersion does not follow a simple self-similarity.

$$\begin{aligned}w_1 \cup w_2 \cup a_1 \cup a_2 \ &= \mathbb{Z}_+,\\ w_1 \cap a_2 = w_2 \cap a_1 &= \emptyset,\end{aligned}$$

For w and a of the clade quartet, all entries of columns are found within the first two columns of the other array, with elements of odd columns found in the first column of the other array and even columns found in the second, leading to (13).

In this sense, a disperses the first two columns of w, or conversely, the first two columns of w constitute a partition of the set s for a in Kimberling's definition for a as a dispersion, and vice versa.

$$\reflectbox{w}_1 \cup \reflectbox{w}_2 \cup \reflectbox{a}_1 \cup \reflectbox{a}_2 = \mathbb{Z}_+.$$

Finally, $\reflectbox{$w$}$ and $\reflectbox{$a$}$ exhibit the same relaxed version of dispersion property (D3), as all entries of a column are found within the first two columns of the other array. An analogy to (D4) is tenuous, however, as these dispersions do not follow a simple self-similarity.

3.6. **Results on interspersion and block decomposition.** For arrays $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, and $\boldsymbol{w}$ the following corollary will describe the block decomposition of rows $N > n$, according to the starting column at which row N begins to intersperse with row n.

Note that the paper does *not* use "blocks" in the matrix-algebra sense of 'non-intersecting submatrices.' Rather, the blocks below are merely sets of consecutive rows in which interspersion with a fixed 'reference row' takes the same form.

Corollary 3.10 (of Proposition 4.6: Block decomposition of rows $N > n$)**.** —

Block decomposition of $\boldsymbol{a}$: Consider row n of $\boldsymbol{a}$. Without loss of generality, strict *interspersion of rows n and $N > n$ takes the form*

$$
\begin{aligned}
&a_{n,k} < a_{N,1} < a_{n,k+1} < a_{N,2} < a_{n,k+2} < \cdots \\
&\qquad \cdots < a_{n,k+h-1} < a_{N,h} < a_{n,k+h} < \cdots .
\end{aligned} \tag{14}
$$

For $n = 2, 3, 4, \ldots$, row n partitions rows $n+1, n+2, \ldots, n+k, \ldots$ into blocks $(a_{n,-1}+1, \ldots, a_{n,0}), (a_{n,0}+1, \ldots, a_{n,1}), \ldots, (a_{n,k-2}+1, \ldots, a_{n,k-1}), \ldots,$ of rows with which row n forms interspersions (14) that begin with, respectively, elements $a_{n,1}, a_{n,2}, \ldots, a_{n,k}, \ldots$.

Rows $n = 0$ and $n = 1$ also intersperse with all subsequent rows, in blocks $(a_{n,0}+1, \ldots, a_{n,1}), (a_{n,1}+1, \ldots, a_{n,2}), \ldots, (a_{n,k-2}+1, \ldots, a_{n,k-1}), \ldots$ of rows whose interspersions begin with, respectively, $a_{n,2}, a_{n,3}, \ldots, a_{n,k}, \ldots$.

Table 13 summarizes the partition of rows $N = n+1, n+2, \ldots$ according to the alignment, k, of their interspersion with row n. In the label for row N, "$k = 1$" indicates that the interspersion (14) of rows n and N begins $a_{n,1} < a_{N,1} < a_{n,2} < \cdots$, "$k = 2$" indicates that the interspersion begins with $a_{n,2} < a_{N,1} < a_{n,3} < \cdots$, and so forth:

$$
\begin{array}{c|ll|cccc}
 & & & \vdots & \vdots & \vdots & \cdots \\
 & N = n & & a_{n,1} & a_{n,2} & a_{n,3} & \cdots \\
\hline
 & N = a_{n,-1}+1, & k = 1 & a_{a_{n,-1}+1,1} & a_{a_{n,-1}+1,2} & a_{a_{n,-1}+1,3} & \cdots \\
a_{n,-2}+1 \text{ rows} & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & N = a_{n,0}, & k = 1 & a_{a_{n,0},1} & a_{a_{n,0},2} & a_{a_{n,0},3} & \cdots \\
\hline
 & N = a_{n,0}+1, & k = 2 & a_{a_{n,0}+1,1} & a_{a_{n,0}+1,2} & a_{a_{n,0}+1,3} & \cdots \\
a_{n,-1}+1 \text{ rows} & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & N = a_{n,1}, & k = 2 & a_{a_{n,1},1} & a_{a_{n,1},2} & a_{a_{n,1},3} & \cdots \\
\hline
 & N = a_{n,1}+1, & k = 3 & a_{a_{n,1}+1,1} & a_{a_{n,1}+1,2} & a_{a_{n,1}+1,3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
a_{n,0}+1 \text{ rows} & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & N = a_{n,2}, & k = 3 & a_{a_{n,2},1} & a_{a_{n,2},2} & a_{a_{n,2},3} & \cdots \\
\hline
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}
$$

TABLE 13. Rows $N \geq n$ of $\boldsymbol{a}$ showing blocks of rows with the same alignment k, upon interspersion (14) of rows N and n

with the exception that the first block of rows $(a_{n,-1}+1, \ldots, a_{n,0})$ is absent for $n = 0, 1$.

Block decomposition of $\boldsymbol{b}$: Similarly for $\boldsymbol{b}$, row n may form a strict interspersion with row $N > n$:

$$
\begin{aligned}
b_{n,k} < b_{N,1} < b_{n,k+1} < b_{N,2} < b_{n,k+2} < \cdots \\
\cdots < b_{n,k+h-1} < b_{N,h} < b_{n,k+h} < \cdots ,
\end{aligned}
\tag{15}
$$

or coincide with row N:

$$
\begin{aligned}
b_{n,k} = b_{N,1} < b_{n,k+1} = b_{N,2} < b_{n,k+2} \leq \cdots \\
\cdots < b_{n,k+h-1} = b_{N,h} < b_{n,k+h} \leq \cdots ,
\end{aligned}
\tag{16}
$$

Table 14 summarizes the partition of rows $N = n+1, n+2, \ldots$ of $\boldsymbol{b}$ according to the alignment, k, of their interspersion with row n:

$$
\begin{array}{c|l l|c c c c}
 & & & \vdots & \vdots & \vdots & \cdots \\
 & \mathbf{N = n} & & \mathbf{b_{n,1}} & \mathbf{b_{n,2}} & \mathbf{b_{n,3}} & \cdots \\
\left.\begin{matrix} b_{n,-2}+1 \\ \text{rows} \end{matrix}\right\{ & N = b_{n,-1}+2, & k=1 & b_{b_{n,-1}+2,1} & b_{b_{n,-1}+2,2} & b_{b_{n,-1}+1,3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & N = b_{n,0} & k=1 & b_{b_{n,0},1} & b_{b_{n,0},2} & b_{b_{n,0},3} & \cdots \\
\hline
 & \mathbf{N = b_{n,0}+1,} & \mathbf{k=2} & \mathbf{b_{b_{n,0}+1,1}} & \mathbf{b_{b_{n,0}+1,2}} & \mathbf{b_{b_{n,0}+1,3}} & \cdots \\
\left.\begin{matrix} b_{n,-1}+1 \\ \text{rows} \end{matrix}\right\{ & N = b_{n,0}+2, & k=2 & b_{b_{n,0}+2,1} & b_{b_{n,0}+2,2} & b_{b_{n,0}+2,3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & N = b_{n,1}, & k=2 & b_{b_{n,1},1} & b_{b_{n,1},2} & b_{b_{n,1},3} & \cdots \\
\hline
 & \mathbf{N = b_{n,1}+1,} & \mathbf{k=3} & \mathbf{b_{b_{n,1}+1,1}} & \mathbf{b_{b_{n,1}+1,2}} & \mathbf{b_{b_{n,1}+1,3}} & \cdots \\
\left.\begin{matrix} b_{n,0}+1 \\ \text{rows} \end{matrix}\right\{ & N = b_{n,1}+2, & k=3 & b_{b_{n,1}+2,1} & b_{b_{n,1}+2,2} & b_{b_{n,1}+2,3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & N = b_{n,2}, & k=3 & b_{b_{n,2},1} & b_{b_{n,2},2} & b_{b_{n,2},3} & \cdots \\
\hline
 & \mathbf{N = b_{n,2}+1,} & \mathbf{k=3} & \mathbf{b_{b_{n,2}+1,1}} & \mathbf{b_{b_{n,2}+1,2}} & \mathbf{b_{b_{n,2}+1,3}} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}
$$

TABLE 14. Rows $N \geq n$ of $\boldsymbol{b}$ showing blocks of rows with the same alignment k (15) upon interspersion or coincidence (16) (boldface) of rows N and n

where coincidence (16) occurs only for rows $N = b_{n,0}+1, b_{n,1}+1, \ldots$, (in boldface), while the interspersion takes the strict form (15) for all other rows.

The exceptions to this general form are the absence of the block of rows $(b_{n,-1}+2, \ldots, b_{n,0})$ for $n = 0, 1$ and of the block of rows $(b_{n,0}+2, \ldots, b_{n,1})$ for $n = 0$, with only the singleton row $b_{n,0}+1 = b_{0,0}+1 = 1$ present.

Block decomposition of $\mathbf{c}$: *Similarly for $\mathbf{c}$, row n may form a strict interspersion with row $N > n$:*

$$
\begin{aligned}
&c_{n,k} < c_{N,1} < c_{n,k+1} < c_{N,2} < c_{n,k+2} < \cdots \\
&\qquad \cdots < c_{n,k+h-1} < c_{N,h} < c_{n,k+h} < \cdots ,
\end{aligned} \tag{17}
$$

or coincide with row N:

$$
\begin{aligned}
&c_{n,k} = c_{N,1} < c_{n,k+1} = c_{N,2} < c_{n,k+2} \leq \cdots \\
&\qquad \cdots < c_{n,k+h-1} = c_{N,h} < c_{n,k+h} \leq \cdots ,
\end{aligned} \tag{18}
$$

Table 15 summarizes the partition of rows $N = n+1, n+2, \ldots$ of $\mathbf{c}$ according to the alignment, k, of their interspersion with row n:

			$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$\mathbf{N = n}$		$\mathbf{c_{n,1}}$	$\mathbf{c_{n,2}}$	$\mathbf{c_{n,3}}$	$\cdots$
	$N = c_{n,-2}+2,$	$k=1$	$c_{c_{n,-2}+2,1}$	$c_{c_{n,-2}+2,2}$	$c_{c_{n,-2}+1,3}$	$\cdots$
$c_{n,-3}$ rows	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$N = c_{n,-1},$	$k=1$	$c_{c_{n,-1},1}$	$c_{c_{n,-1},2}$	$c_{c_{n,-1},3}$	$\cdots$
	$N = c_{n,-1}+1,$	$k=2$	$c_{c_{n,-1}+1,1}$	$c_{c_{n,-1}+1,2}$	$c_{c_{n,-1}+1,3}$	$\cdots$
$c_{n,-2}+1$ rows	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$N = c_{n,0},$	$k=2$	$c_{c_{n,0},1}$	$c_{c_{n,0},2}$	$c_{c_{n,0},3}$	$\cdots$
	$\mathbf{N = c_{n,0}+1,}$	$\mathbf{k=3}$	$\mathbf{c_{c_{n,0}+1,1}}$	$\mathbf{c_{c_{n,0}+1,2}}$	$\mathbf{c_{c_{n,0}+1,3}}$	$\cdots$
	$N = c_{n,0}+2,$	$k=3$	$c_{b_{n,0}+2,1}$	$c_{c_{n,0}+2,2}$	$c_{c_{n,0}+2,3}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$c_{n,-1}$ rows	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$N = c_{n,1},$	$k=3$	$c_{c_{n,1},1}$	$c_{c_{n,1},2}$	$c_{c_{n,1},3}$	$\cdots$
	$N = c_{n,1}+1,$	$k=4$	$c_{c_{n,1}+1,1}$	$c_{c_{n,1}+1,2}$	$c_{c_{n,1}+1,3}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$c_{n,0}+1$ rows	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
	$N = c_{n,2},$	$k=4$	$c_{c_{n,2},1}$	$c_{c_{n,2},2}$	$c_{c_{n,2},3}$	$\cdots$
	$\mathbf{N = c_{n,2}+1,}$	$\mathbf{k=5}$	$\mathbf{c_{c_{n,2}+1,1}}$	$\mathbf{c_{c_{n,2}+1,2}}$	$\mathbf{c_{c_{n,2}+1,3}}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

TABLE 15. Rows $N \geq n$ of $\boldsymbol{c}$ showing blocks of rows with the same alignment k, (17) upon interspersion or coincidence (18) (boldface) of rows N and n

where coincidence (18) occurs only for rows $N = c_{n,0}+1, c_{n,2}+1, \ldots$, *(in boldface), while the interspersion takes the strict form (17) for all other rows.*

As exceptions to this general form, for $n = 0$, *blocks* $(c_{n,-2}+2, \ldots, c_{n,-1})$,$(c_{n,-1}+2, \ldots, c_{n,0})$, *and* $(c_{n,0}+2, \ldots, c_{n,1})$ *will be absent, with only the singleton row* $c_{n,0}+1 = c_{0,0}+1 = 1$ *present, while for* $n = 1$, *block* $(c_{n,-2}+2, \ldots, c_{n,-1})$ *will be absent.*

Block decomposition of $\boldsymbol{d}$: *Finally, for* $\boldsymbol{d}$, *row* n *may form a strict interspersion with row* $N > n$:

$$
\begin{aligned}
d_{n,k} < d_{N,1} < d_{n,k+1} < d_{N,2} < d_{n,k+2} < \cdots \\
\cdots < d_{n,k+h-1} < d_{N,h} < d_{n,k+h} < \cdots ,
\end{aligned} \tag{19}
$$

Table 16 summarizes the partition of rows $N = n+1, n+2, \ldots$ *of* $\boldsymbol{d}$ *according to the alignment,* k, *of their interspersion with row* n:

$$
\begin{array}{cl|cccc}
 & & \vdots & \vdots & \vdots & \ldots \\
 & N = n & d_{n,1} & d_{n,2} & d_{n,3} & \ldots \\
\hline
\begin{array}{c} d_{n,-3}+2 \\ \text{rows} \end{array} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{l} N = d_{n,-2}+2, k=1 \\ \vdots \qquad\qquad \vdots \\ N = d_{n,-1}+1, k=1 \end{array} & \begin{array}{c} d_{d_{n,-2}+2,1} \\ \vdots \\ d_{d_{n,-1}+1,1} \end{array} & \begin{array}{c} d_{d_{n,-2}+2,2} \\ \vdots \\ d_{d_{n,-1}+1,2} \end{array} & \begin{array}{c} d_{d_{n,-2}+2,3} \\ \vdots \\ d_{d_{n,-1}+1,3} \end{array} & \begin{array}{c} \ldots \\ \ldots \\ \ldots \end{array} \\
\hline
\begin{array}{c} d_{n,-2}+2 \\ \text{rows} \end{array} \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. & \begin{array}{l} N = d_{n,-1}+2, k=2 \\ \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \\ N = d_{n,0}+1, \ k=2 \end{array} & \begin{array}{c} d_{d_{n,-1}+2,1} \\ \vdots \\ \vdots \\ d_{d_{n,0}+1,1} \end{array} & \begin{array}{c} d_{d_{n,-1}+2,2} \\ \vdots \\ \vdots \\ d_{d_{n,0}+1,2} \end{array} & \begin{array}{c} d_{d_{n,-1}+2,3} \\ \vdots \\ \vdots \\ d_{d_{n,0}+1,3} \end{array} & \begin{array}{c} \ldots \\ \ldots \\ \ldots \\ \ldots \end{array} \\
\hline
\begin{array}{c} d_{n,-1}+2 \\ \text{rows} \end{array} \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. & \begin{array}{l} N = d_{n,0}+2, \ k=3 \\ \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \\ N = d_{n,1}+1, \ k=3 \end{array} & \begin{array}{c} d_{d_{n,0}+2,1} \\ \vdots \\ \vdots \\ \vdots \\ d_{d_{n,1}+1,1} \end{array} & \begin{array}{c} d_{d_{n,0}+2,2} \\ \vdots \\ \vdots \\ \vdots \\ d_{d_{n,1}+1,2} \end{array} & \begin{array}{c} d_{d_{n,0}+2,3} \\ \vdots \\ \vdots \\ \vdots \\ d_{d_{n,1}+1,3} \end{array} & \begin{array}{c} \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \end{array} \\
\hline
 & \vdots \qquad\qquad \vdots & \vdots & \vdots & \vdots & \ldots
\end{array}
$$

TABLE 16. Rows $N \geq n$ of $\boldsymbol{d}$ showing blocks of rows with the same alignment k, upon interspersion (19) of rows N and n

with the exception that the first block of rows $(d_{n,-2}+2, \ldots, d_{n,-1}+1)$ *is absent for* $n = 0$.

Block decomposition of $\boldsymbol{w}$: *Consider row* n *of* $\boldsymbol{w}$. *Without loss of generality,* strict *interspersion of rows* n *and* $N > n$ *takes the form*

$$
\begin{aligned}
w_{n,k} < w_{N,1} < w_{n,k+1} < w_{N,2} < w_{n,k+2} < \cdots \\
\cdots < w_{n,k+h-1} < w_{N,h} < w_{n,k+h} < \cdots .
\end{aligned} \tag{20}
$$

For $n = 1, 2, \ldots, k, \ldots$, *row* n *partitions rows* $n+1, n+2, \ldots, n+k, \ldots$ *into blocks* $(w_{n,-1}+1, \ldots, w_{n,0}-1), (w_{n,0}, \ldots, w_{n,1}-1), \ldots, (w_{n,k-2}, \ldots, w_{n,k-1}-1), \ldots$, *of rows with which row* n *forms interspersions (14) that begin with, respectively, elements* $w_{n,1}, w_{n,2}, \ldots, w_{n,k}, \ldots$.

Row $n = 0$ also intersperses with all subsequent rows, in blocks $(w_{n,1}, \ldots, w_{n,2} - 1), \ldots, (w_{n,k-2}, \ldots, w_{n,k-1} - 1), \ldots$ of rows whose interspersions begin with, respectively, $w_{n,3}, w_{n,4}, \ldots, w_{n,k}, \ldots$.

Table 13 summarizes the partition of rows $N = n+1, n+2, \ldots$ according to the alignment, k, of their interspersion with row n. In the label for row N, "$k = 1$" indicates that the interspersion (14) of rows n and N begins $w_{n,1} < w_{N,1} < w_{n,2} < \cdots$, "$k = 2$" indicates that the interspersion begins with $w_{n,2} < w_{N,1} < w_{n,3} < \cdots$, and so forth:

			$\vdots$	$\vdots$	$\vdots$	$\ldots$
	$N = n$		$w_{n,1}$	$w_{n,2}$	$w_{n,3}$	$\ldots$
$w_{n,-2} - 1$ rows	$N = w_{n,-1} + 1,$	$k = 1$	$w_{w_{n,-1}+1,1}$	$w_{w_{n,-1}+1,2}$	$w_{a_{n,-1}+1,3}$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
	$N = w_{n,0} - 1,$	$k = 1$	$w_{w_{n,0}-1,1}$	$w_{w_{n,0}-1,2}$	$w_{w_{n,0}-1,3}$	$\ldots$
$w_{n,-1}$ rows	$N = w_{n,0},$	$k = 2$	$w_{w_{n,0},1}$	$w_{w_{n,0},2}$	$w_{w_{n,0},3}$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
	$N = w_{n,1} - 1,$	$k = 2$	$w_{w_{n,1}-1,1}$	$w_{w_{n,1}-1,2}$	$w_{w_{n,1}-1,3}$	$\ldots$
$w_{n,0}$ rows	$N = w_{n,1},$	$k = 3$	$w_{w_{n,1},1}$	$w_{w_{n,1},2}$	$w_{w_{n,1},3}$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
	$N = w_{n,2} - 1,$	$k = 3$	$w_{w_{n,2}-1,1}$	$w_{w_{n,2}-1,2}$	$w_{w_{n,2}-1,3}$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

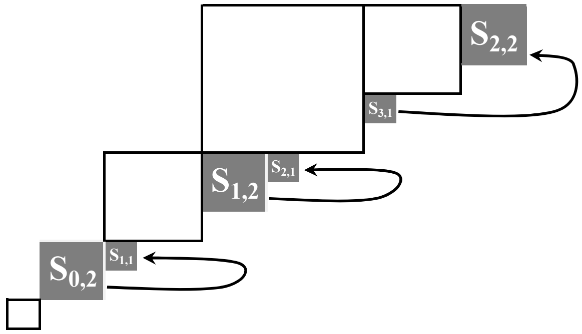
TABLE 17. Rows $N \geq n$ of $\boldsymbol{w}$ showing blocks of rows with the same alignment k, upon interspersion (14) of rows N and n

with the exception that the first two blocks of rows $(w_{n,-1} + 1, \ldots, w_{n,0} - 1)$ and $(w_{n,0}, \ldots, w_{n,1} - 1)$ are absent for $n = 0$.

Proof. Tables 13–17 show the (starting and ending) row indices for each block inside the braces to the immediate left of the arrays, and indicate the number of rows per block at the far left, just outside the braces.

Calculate $a_{n,-1} + 1 = b_{n,-1} + 2 = c_{n,-2} + 2 = d_{n,-2} + 2 = w_{n,-1} + 1 = n + 1$ using (1)–(4) and the formula for the Wythoff array in Table 12 to confirm that the row immediately following row n in each table is indeed row $n+1$. Where this row and others are missing in the exceptions noted, confirm that the starting index of the missing block is greater than the ending index, so that the block contains no rows, or equivalently, that the number of rows shown outside the braces evaluates to zero.

The remainder of the proof for $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$, follows from Propositions 4.6–4.9, respectively. Similarly, the proof for $\boldsymbol{w}$ follows from Proposition 4.10. □



1	2	4	7
3	5	9	15
6	10	17	28
8	13	22	36

FIGURE 2. Corollary 3.10 on interspersion of rows: Initial rows of $\boldsymbol{a}$ shown in the quilt (Figure 1), at left, and in the array itself (Table 1), at right

Remark 3.3. Corollary 3.10 considered interspersion / interspersoid arrays that satisfy certain requirements. Firstly, the array can be expressed in a closed form allowing its extension to columns $k = 0, -1, -2, \ldots$, one of which gives the sequence of positive integers (see Tables 1–8). This allows the expression of the row index $n+1$ in each of the tables in simple terms of an element in row n of the extended array. As the proof states, $a_{n,-1}+1 = b_{n,-1}+2 = c_{n,-2}+2 = d_{n,-2}+2 = w_{n,-1}+1 = n+1$.

In order for its columns to extrapolate backwards to the positive integers, the array must be weighted more toward row growth than toward column growth. For example, the initial columns of $\mathfrak{w}$ and $\mathfrak{d}$ grow so slowly that the 0^{th} column by extension does not increase strictly.

Secondly, the arrays break into blocks in such a way that the block heights can be expressed simply in terms of successive elements of row n and further, the arrays follow a row recurrence such that the partial sums of these block heights can be expressed generally by the same simple formula in successive elements of row n. The latter allows the sequences of starting and ending rows of the blocks also to be written using a simple formula in successive elements of row n.

This is not the case for all interspersion / interspersoid arrays. Again, for $\mathfrak{w}$ and $\mathfrak{d}$, block heights can indeed be expressed simply in terms of successive elements of row n. However, the partial sums of these block heights do not take a simple form in successive elements of row n.

Remark 3.4. Figure 1 allows Corollary 3.10 to be visualized by interpreting "$<$" in (14) as the relation "south of" for black quilt squares, and for $\boldsymbol{c}$ by interpreting "$<$" in (17) as the relation "west of." The arrows in Figure 2, at left, illustrates the relation "$<$" for $\boldsymbol{a}$.

For example, the interspersion of rows 0 and 1 of $\boldsymbol{a}$ does *not* begin $a_{0,1} < a_{1,1} < a_{0,2} < \cdots$, with alignment $k = 1$, but rather $a_{0,2} < a_{1,1} < a_{0,3} < \cdots$, with alignment $k = 2$, whereas $S_{0,2}$, the zeroth square of size 2 (the 2×2 on the main diagonal) lies south of $S_{1,1}$, the first off-diagonal square of size 1.

The same is true for the interspersion of rows 1 and 2 of $\boldsymbol{a}$. It begins $a_{1,2} < a_{2,1} < a_{1,3} < \cdots$, again with alignment $k = 2$ rather than $k = 1$, as $S_{1,2}$, the first off-diagonal square of size 2, lies south of $S_{2,1}$, the second off-diagonal square of size 1.

However, the interspersion of rows 2 and 3 of $\boldsymbol{a}$, *does* begin $a_{2,1} < a_{3,1} < a_{2,2} < \cdots$, with alignment $k = 1$, whereas $S_{2,2}$, the second off-diagonal square of size 2 lies *north* of $S_{3,1}$, the third off-diagonal square of size 1. This shows why the first block of Table 13 is missing for $n = 0, 1$, but present for $n \geq 2$. More generally, then, for $n = 0, 1$, we have $a_{n,k+1} < a_{n+1,k}$, whereas for $n \geq 2$, $a_{n+1,k} < a_{n,k+1}$. When considering each antidiagonal of $\boldsymbol{a}$ as a sequence, the 1st and 2nd antidiagonals are monotonic, whereas the 3rd and subsequent antidiagonals are all non-monotonic, as shown by the arrows in Figure 2, at right.

Corollary 3.11 (of Proposition 4.6: Interspersive properties of the second block)**.** *For the quilt black arrays* $\boldsymbol{a}$*,* $\boldsymbol{b}$*,* $\boldsymbol{c}$ *and* $\boldsymbol{d}$*, consider rows* $n \geq 0$*; for* $\boldsymbol{\beta}$*,* $\boldsymbol{\gamma}$ *and* $\boldsymbol{w}$*, consider rows* $n \geq 1$*; and for* $\boldsymbol{\alpha}$*, consider rows* $n \geq 2$*.*

Then, interspersion properties (i) and (ii), below, describe the interspersion of row n *with rows at the top and bottom, respectively, of the second block below row* n *induced by interspersion. For* $\boldsymbol{a}$*,* $\boldsymbol{b}$*,* $\boldsymbol{c}$*,* $\boldsymbol{d}$ *and* $\boldsymbol{w}$ *in particular, Tables 13 through 17 show this block (refer to Corollary 3.10).*

(i) The first property describes the interspersion of row n *with the top row of the second block below row* n*:*

$$\begin{aligned}
a_{\kappa(n)+1,k-2} &< a_{n,k} < a_{\kappa(n)+1,k-1}, \text{ in particular } a_{n,k} = a_{\kappa(n)+1,k-1} - F_{k-1};\\
b_{\kappa(n)+1,k-2} &< b_{n,k} = b_{\kappa(n)+1,k-1}, \text{ in particular } b_{n,k} = b_{\kappa(n)+1,k-1};\\
c_{\kappa(n)+1,k-2} &\leq c_{n,k} < c_{\kappa(n)+1,k-1}, \text{ in particular } c_{n,k} = c_{\kappa(n)+1,k-1} - F_{k+1};\\
d_{\kappa(n)+1,k-2} &< d_{n,k} < d_{\kappa(n)+1,k-1}, \text{ in particular } d_{n,k} = d_{\kappa(n)+1,k-1} - F_{k};\\
w_{\kappa(n+1),k-2} &< w_{n,k} < w_{\kappa(n+1),k-1}, \text{ in particular } w_{n,k} = w_{\kappa(n+1),k-1} - 2F_{k}.
\end{aligned}$$

Thus, each column of $\boldsymbol{b}$ *contains its successor, that is,* $b_k \supset b_{k+1}$*,* $k = 1, 2, 3, \ldots$*. Furthermore,*

$$\begin{aligned}
\alpha_{\kappa(n),k-2} &\leq \alpha_{n,k} = \alpha_{\kappa(n),k-1}, \quad \text{ in particular } \alpha_{n,k} = \alpha_{\kappa(n),k-1};\\
\beta_{\kappa(n)+1,k-2} &\leq \beta_{n,k} < \beta_{\kappa(n)+1,k-1}, \text{ in particular } \beta_{n,k} = \beta_{\kappa(n)+1,k-1} - F_{k};\\
\gamma_{\kappa(n)+1,k-2} &< \gamma_{n,k} < \gamma_{\kappa(n)+1,k-1}, \text{ in particular } \gamma_{n,k} = \gamma_{\kappa(n)+1,k-1} - F_{k};\\
\delta_{\kappa(n)+1,k-2} &< \delta_{n,k} = \delta_{\kappa(n)+1,k-1}, \text{ in particular } \delta_{n,k} = \delta_{\kappa(n)+1,k-1}.
\end{aligned}$$

Thus, arrays $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$ *have the same column containment property as* $\boldsymbol{b}$*, that is,* $\alpha_k \supset \alpha_{k+1}$*, and* $\delta_k \supset \delta_{k+1}$*,* $k = 1, 2, 3, \ldots$*.*

(ii) The second property describes the interspersion of row n *with the bottom row of the second block below row* n*:*

$$\begin{aligned}
a_{\lambda(n)+1,k-2} &< a_{n,k} < a_{\lambda(n)+1,k-1}, \text{ in particular } a_{n,k} = a_{\lambda(n)+1,k-2} + F_{k-1};\\
b_{\lambda(n)+2,k-2} &= b_{n,k} < b_{\lambda(n)+2,k-1}, \text{ in particular } b_{n,k} = b_{\lambda(n)+2,k-2};\\
c_{\lambda(n)+1,k-2} &= c_{n,k} < c_{\lambda(n)+1,k-1}, \text{ in particular } c_{n,k} = c_{\lambda(n)+1,k-2};\\
d_{\lambda(n)+1,k-2} &< d_{n,k} < d_{\lambda(n)+1,k-1}, \text{ in particular } d_{n,k} = d_{\lambda(n)+1,k-2} + F_{k};\\
w_{\lambda(n+1)-2,k-2} &< w_{n,k} < w_{\lambda(n+1)-2,k-1}, \text{ in particular } w_{n,k} = w_{\lambda(n+1)-2,k-2} + F_{k}.
\end{aligned}$$

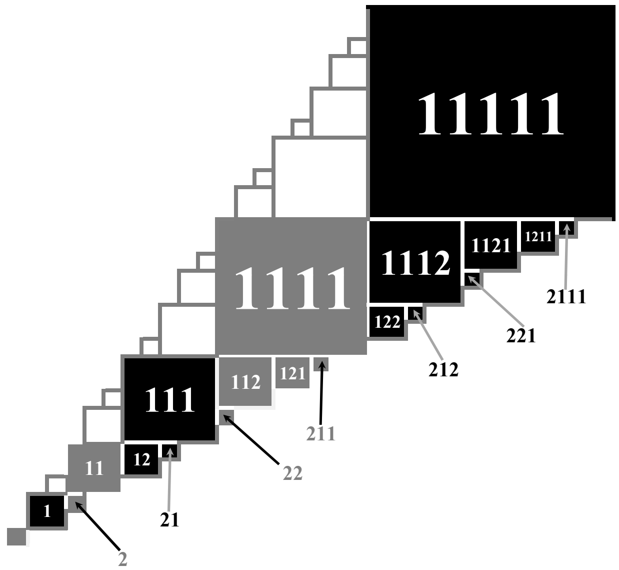


FIGURE 3. Compositions using only 1s and 2s in the quilt (Figure 1) [6]

Thus, each column of $\mathbf{c}$ *contains its second successor, that is,* $c_k \supset c_{k+2}$, $k = 1, 2, 3, \ldots$. *Furthermore,*

$$\begin{aligned}
\alpha_{\lambda(n)-1,k-2} &= \alpha_{n,k} \leq \alpha_{\lambda(n)-1,k-1}, \text{ in particular } \alpha_{n,k} = \alpha_{\lambda(n)-1,k-2};\\
\beta_{\lambda(n),k-2} &= \beta_{n,k} < \beta_{\lambda(n),k-1}, \quad \text{ in particular } \beta_{n,k} = \beta_{\lambda(n),k-2};\\
\gamma_{\lambda(n)+1,k-2} &= \gamma_{n,k} < \gamma_{\lambda(n)+1,k-1}, \text{ in particular } \gamma_{n,k} = \gamma_{\lambda(n)+1,k-2};\\
\delta_{\lambda(n)+2,k-2} &= \delta_{n,k} < \delta_{\lambda(n)+2,k-1}, \text{ in particular } \delta_{n,k} = \delta_{\lambda(n)+2,k-2}.
\end{aligned}$$

Thus, arrays $\boldsymbol{\beta}$ *and* $\boldsymbol{\gamma}$ *have the same column containment property as* $\mathbf{c}$*, that is,* $\beta_k \supset \beta_{k+2}$, *and* $\gamma_k \supset \gamma_{k+2}$, $k = 1, 2, 3, \ldots$.

Proof. From (1)–(4) calculate that $\kappa(n) + 1 = a_{n,0} + 1 = b_{n,0} + 1 = c_{n,-1} + 1 = d_{n,-1} + 2$, and that $\lambda(n) + 1 = a_{n,1} = b_{n,1} = c_{n,0} + 1 = d_{n,0} + 1$, confirming that interspersion properties (i) and (ii) indeed treat the top, respectively, bottom of the second block below row n in Tables 13 through 16. In Table 17 consider that $\kappa(n + 1) = w_{n,0}$ and $\lambda(n + 1) - 2 = w_{n,1} - 1$. The remainder follows from Corollary 3.10, with similar arguments made for the quilt white arrays. □

Remark 3.5. In particular, Corollary 3.11 identifies seven cases in which a row of a quilt array is a shifted version of the row above it, namely: $b_{1,k-1} = b_{0,k}$, $b_{2,k-1} = b_{1,k}$, and $c_{1,k-2} = c_{0,k}$, these first three already noted in Corollary 3.10, as well as $\alpha_{3,k-1} = \alpha_{2,k}$, $\alpha_{4,k-1} = \alpha_{3,k}$, $\beta_{2,k-2} = \beta_{1,k}$, and $\delta_{2,k-1} = \delta_{1,k}$.

3.7. **Results on quilt alignment and complementarity.** Part 2 [6] showed that the quilt's black squares provide visualization of restricted compositions of integers (those using only twos and ones), with square $S_{n,k}$ corresponding to a restricted composition of $F^{-1}(n) + k - 1$. That is, spinal square $S_{0,F^{-1}(n)+k}$ and all black quilt squares directly south of it graphically illustrate all 2–1-compositions of $F^{-1}(n) + k - 1$ (Figure 3).

The quilt also serves as an abacus for certain *complementary equations* studied by Kimberling [3], [4], and discussed in Part 2 of this paper [6]. These complementary equations involve specific columns columns, a_k^+, b_k^+, c_k^+, and d_k^+ of the quilt black arrays, and α_k, β_k, γ_k, and δ_k of the quilt white arrays. The symmetry of the quilt about the diagonal allows any pair of quilt sequences to be graphically aligned, thus allowing visualization of the identities.

	S_1	p
a_k^+	$2F_{k+2}-1$	$k+1$
b_k^+	$F_{k+4}-2$	$k+1$
c_k^+	$F_{k+4}-1$	$k+2$
d_k^+	$2F_{k+3}-2$	$k+2$
α_k	1	k
β_k	F_{k+1}	k
γ_k	$F_{k+3}-1$	$k+1$
δ_k	$F_{k+4}-2$	$k+1$

Table 18. Parameters S_1 and p for the cohort form of the quilt arrays by column k

Part 2 [6] categorized certain integer sequences S_n as *Fibonacci cohort sequences from the 1*st *cohort* and expressed a recurrence for these sequences. It characterized such sequences by two parameters: The parameter S_1 specifies the initial value at which the recursion for S_n begins and the parameter p specifies how quickly it spreads out. Table 18 shows the parameters S_1 and p for Fibonacci cohort sequences related to the quilt, employed by their *cohort forms* and *pure-κ forms* given in (1) through (8).

p	1	2	3	4	5	6	7	8
$\alpha_p\equiv$	κ	κ^2	κ^3	κ^4	κ^5	κ^6	κ^7	κ^8
$\beta_p\equiv$	κ	λ	$\kappa\lambda$	λ^2	$\kappa\lambda^2$	λ^3	$\kappa\lambda^3$	λ^4
α_p	1	1	1	1	1	1	1	1
β_p	1	2	3	5	13	21	34	55
γ_{p-1}		2	4	7	12	20	33	54
c_{p-2}^+			4	7	12	20	33	54
d_{p-2}^+			4	8	14	24	40	66
a_{p-1}^+		3	5	9	15	25	41	67
$b_{p-1}^+=\delta_{p-1}$		3	6	11	19	32	53	87

Table 19. S_1 vs. p for Columns of Quilt Arrays

Part 2 of the paper considered three cannonical forms for Fibonacci cohort sequences. The cohort form $S_n = F_p\kappa(n) + F_{p-1}n - F_{p+1} + S_1$ employed an affine combination of $\kappa(n)$ and n. The Wythoff-composition form was homogeneous, writing S as the representative of an S_1-class of Wythoff compositions in $\{\kappa, \lambda\}^\star$, and

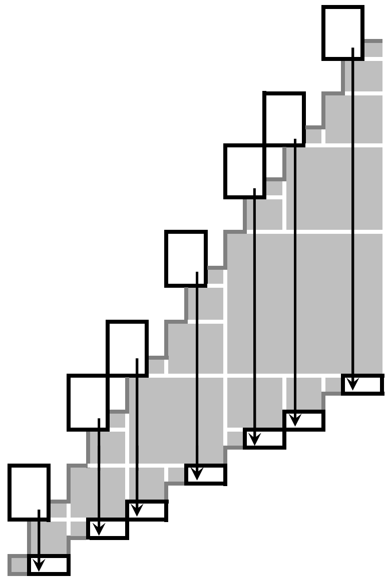


FIGURE 4. Correspondence of β_2 and γ_1 in the quilt (21)

then matching p by initial applications of κ to the argument n of S. That is, writing S as either $\kappa^\star$, where $\kappa^\star(n) = \lfloor\lfloor n\phi\rfloor\cdots\phi\rfloor$, or as a member of a class modulo $\kappa^\star$, where the standard class representatives were ordered by increasing values of S_1, as $I, \lambda, \kappa\lambda, \kappa^2\lambda, \lambda^2, \kappa^3\lambda, \lambda\kappa\lambda, \kappa\lambda^2, \ldots$. Finally, the pure-$\kappa$ form $S_n = \kappa^p(n) + (S_1 - 1)$, used a nested iteration of the Wythoff function κ, plus a constant. In this context, the cohort form and homogeneous Wythoff composition forms proved to be equivalent, whereas the pure-κ form gave identical values on the positive integers, though not necessarily giving the same value at zero.

Next, reconsider the quilt black arrays $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, and quilt white arrays $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, for $n \geq 1$, treating each column of each array as a distinct sequence $S = S_1, S_2, \ldots, S_k, \ldots$. Table 19 gives specific values of S_1 versus p for sequences a_k^+, b_k^+, c_k^+, d_k^+, α_k, β_k, γ_k, and δ_k, as well as the pure-κ forms for columns of $\boldsymbol{\alpha}$ and the Wythoff-composition forms for columns of $\boldsymbol{\beta}$. Specifically, Part 2 [6] showed that $\beta_{n,k} = \begin{cases} \lambda^{k/2}(n), & k \text{ even;} \\ \kappa\lambda^{(k-1)/2}(n), & k \text{ odd.} \end{cases}$

By definition, the dimensions of 1×1 quilt squares and 1×2 quilt rectangles, ensure the column equivalences $a_1 = b_1$ and $c_1 = d_1$, respectively $\alpha_1 = \beta_1$. Selecting p so that values of S_1 match in Table 19 yields additional equivalences and of quilt columns.

3.7.1. *Quilt alignment and correspondences.* Before proceeding the graphical complementarity in the quilt, first consider graphical correspondences, that is, individual columns of the different arrays of quilt sequences (Tables 1–8) that coincide with one another.

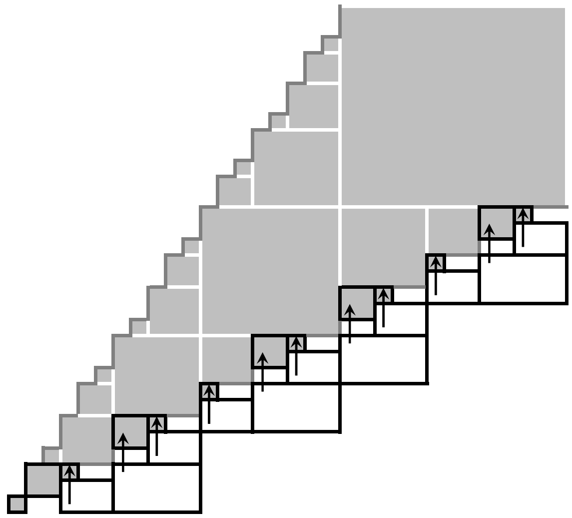


FIGURE 5. Correspondence of γ_2 with c_1^+ and γ_3 with c_2^+ in the quilt (22), (23)

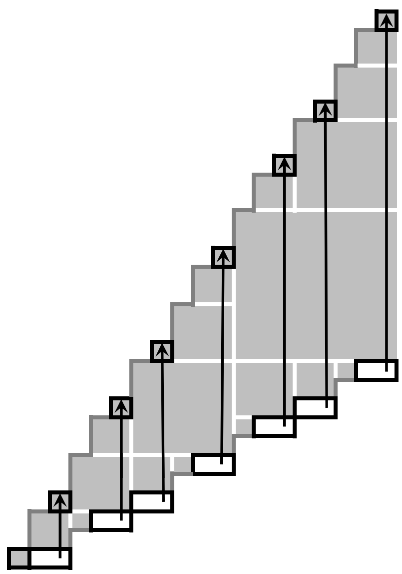


FIGURE 6. Correspondence of δ_1 and a_1^+ in the quilt (24)

Proposition 3.12 (Correspondences in the quilt, follow from (1)–(8))**.**

(21) $\beta_2 = \gamma_1$ *(Figure 4);*

$\gamma_{k+1} = c_k^+, k \geq 1,$ *in particular:*

(22) $\gamma_2 = c_1^+ = d_1^+$ *(Figure 5),*

(23) $\gamma_3 = c_2^+$ *(Figure 5);*

$\delta_k = b_k^+, k \geq 1,$ *in particular:*

(24) $\delta_1 = a_1^+ = b_1^+$ *(Figure 6).*

3.7.2. *Quilt alignment and complementarity.* Selecting p so that values of S_1 match in Table 19 also yields hierarchies of quilt columns, shown in Figures 7 through 12.

Consider that the complementary lower and upper Wythoff sequences $K = \kappa(\mathbb{Z}_{\geq 1})$ and $\Lambda = \lambda(\mathbb{Z}_{\geq 1})$, respectively, partition the positive integers $\mathbb{Z}_{\geq 1}$. The partition shown by the binary trees in Figure 7 repeats this principle, further partitioning K into K^2 and $K\Lambda$. Matching the parameter pairs (p, S_1) with Table 19, it follows that the complementary quilt sequences shown in Figure 7 partition the positive integers $\mathbb{Z}_{\geq 1}$.

Now, to extend this idea to $\mathbb{Z}_{\geq 2}$, partition the integers $2, 3, 4\ldots$ into $K+1$ and $\Lambda+1 = a_1^+ = b_1^+ = \delta_1$, where $K+1$ partitions further into $K^2+1 = \beta_2 = \gamma_1$ and $K\Lambda+1 = c_1^+ = d_1^+ = \gamma_2$, to arrive at the partition of $\mathbb{Z}_{\geq 2}$ shown in Figure 8. Thus, $\big\{\{1\}, \{a_1^+ = b_1^+ = \delta_1\}, \{\beta_2 = \gamma_1\}, \{c_1^+ = d_1^+ = \gamma_2\}\big\}$ is a partition of $\mathbb{Z}_{\geq 1}$. Figures 8, 9, 10, and 11 extend the idea to $\mathbb{Z}_{\geq 2}$, $\mathbb{Z}_{\geq 3}$, $\mathbb{Z}_{\geq 4}$, and $\mathbb{Z}_{\geq 5}$, respectively. Figure 12 combines the results, for which Propositions 3.12 and 3.13 provide visualizations in the quilt.

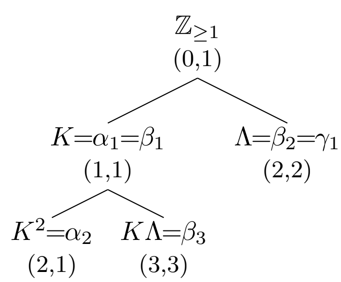


FIGURE 7. Partition of $\mathbb{Z}_{\geq 1}$ using complementary quilt sequences with parameters (p, S_1)

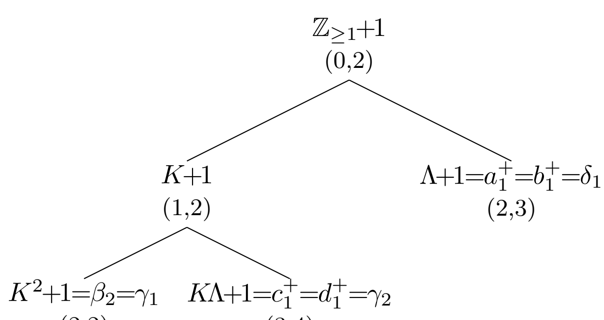


FIGURE 8. Partition of $\mathbb{Z}_{\geq 2}$ using complementary quilt sequences with parameters (p, S_1). Figure 13 shows this complementarity in the quilt.

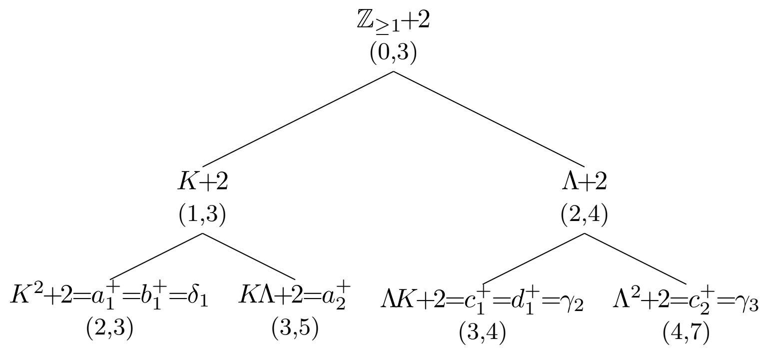


FIGURE 9. Partition of $\mathbb{Z}_{\geq 3}$ using complementary quilt sequences with parameters (p, S_1). Figure 14 shows this complementarity in the quilt.

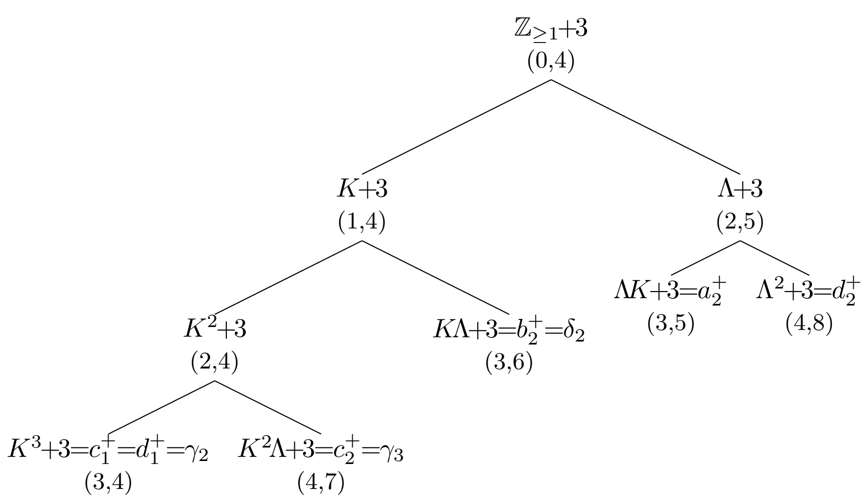


FIGURE 10. Partition of $\mathbb{Z}_{\geq 4}$ using complementary quilt sequences with parameters (p, S_1)

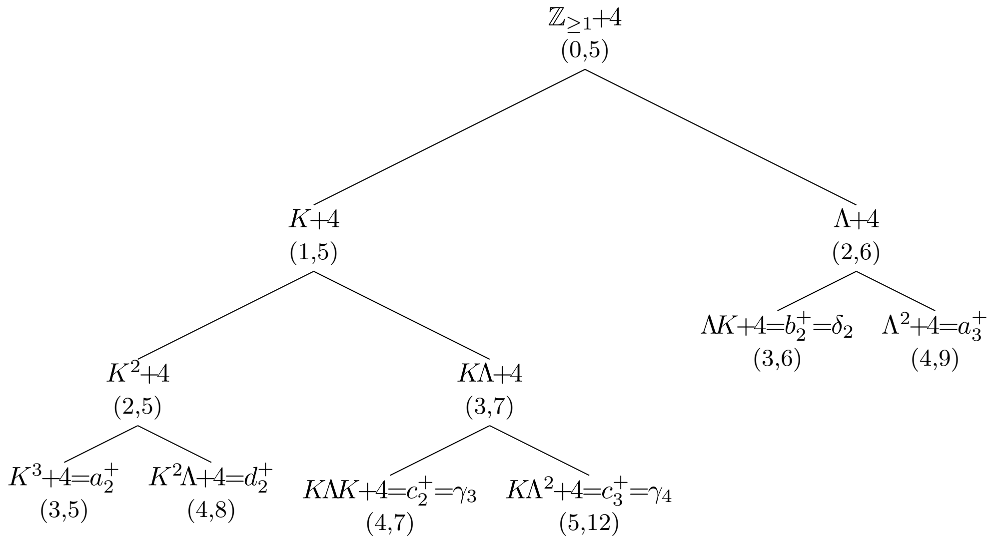


FIGURE 11. Partition of $\mathbb{Z}_{\geq 5}$ using complementary quilt sequences with parameters (p, S_1)

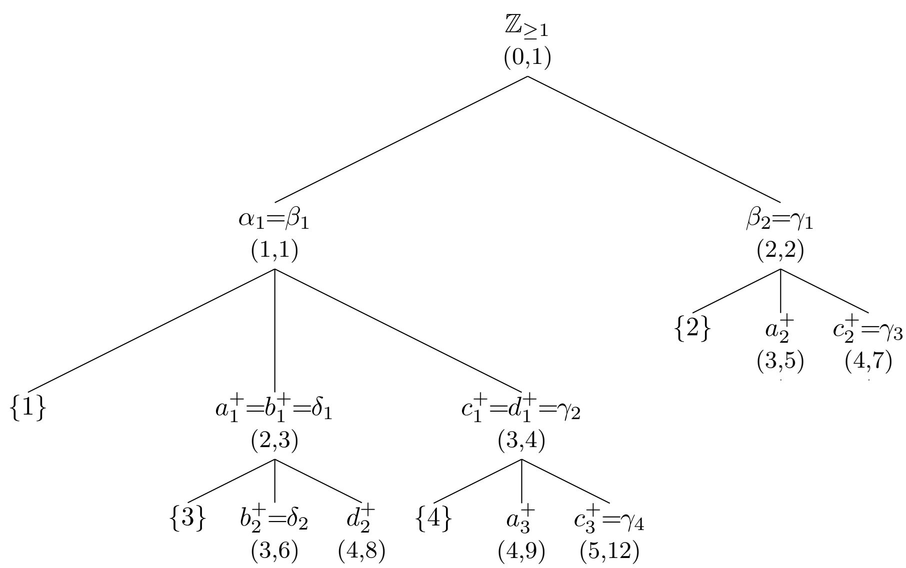


FIGURE 12. Partition of $\mathbb{Z}_{\geq 1}$ using complementary quilt sequences with parameters (p, S_1). Figure 15 shows this complementarity in the quilt.

Proposition 3.13 (Complementarity in the quilt)**.** *Figures 7 through 12 indicate various complementary quilt sequences, among these:*

$$a_1^+ \cup \gamma_1 \cup \gamma_2 = \mathbb{Z}_{\geq 2}, \quad \textit{(Figure 8)}, \qquad \gamma_1 \cap \gamma_2 = \gamma_1 \cap a_1^+ = \gamma_2 \cap a_1^+ = \varnothing; \tag{25}$$

$$a_1^+ \cup a_2^+ \cup c_1^+ \cup c_2^+ = \mathbb{Z}_{\geq 3}, \quad \textit{(Figure 9)}, \tag{26}$$

$$a_1^+ \cap a_2^+ = a_1^+ \cap c_1^+ = a_1^+ \cap c_2^+ = a_2^+ \cap c_1^+ = a_2^+ \cap c_2^+ = c_1^+ \cap c_2^+ = \varnothing;$$

$$\{1\} \cup b_1^+ \cup d_1^+ = K, \quad \textit{(Figure 12)}, \qquad b_1^+ \cap d_1^+ = \varnothing. \tag{27}$$

Consequently,

$$\bigcup_{n=0}^{\infty} \bigcup_{k=2}^{\infty} a_{n,k} = \gamma_1 \cup \gamma_2 = \bigcup_{k=1}^{\infty} \gamma_k \ \textit{(Figure 13)}; \tag{28}$$

$$\bigcup_{n=0}^{\infty} \bigcup_{k=3}^{\infty} a_{n,k} = c_1^+ \cup c_2^+ = \bigcup_{k=1}^{\infty} c_k^+ \ \textit{(Figure 14)}; \tag{29}$$

$$\bigcup_{n=0}^{\infty} \bigcup_{k=2}^{\infty} d_{n,k} = \quad b_1^+ \quad = \bigcup_{k=1}^{\infty} b_k^+ \ \textit{(Figure 15)}. \tag{30}$$

Proof. It follows from partition (25) that $\{1\} \cup a_1^+ \cup \gamma_1 \cup \gamma_2 = \mathbb{Z}_{\geq 1}$, with $1 \notin a_1^+ \cup \gamma_1 \cup \gamma_2$. By Proposition 3.6, $\bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} a_{n,k} = \mathbb{Z}_{\geq 1}$, demonstrating the first equality in (28). Corollary 3.11 shows that $\gamma_1 \supset \gamma_3 \supset \gamma_5 \supset \ldots$ and $\gamma_2 \supset \gamma_4 \supset \gamma_6 \supset \ldots$ demonstrating the second equality in (28).

It follows from partition (26) that $\{1,2\} \cup a_1^+ \cup a_2^+ \cup c_1^+ \cup c_2^+ = \mathbb{Z}_{\geq 1}$ with $1, 2 \notin a_1^+ \cup a_2^+ \cup c_1^+ \cup c_2^+$. By Proposition 3.6, $\bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} a_{n,k} = \mathbb{Z}_{\geq 1}$, demonstrating the first equality in (29). To demonstrate the second, consider from Corollary 3.11 that $c_1 \supset c_3 \supset c_5 \supset \ldots$ and $c_2 \supset c_4 \supset c_6 \supset \ldots$. As shown in Corollary 3.10 and cited in Remark 3.5, $c_{1,k-2} = c_{0,k}$, for $k \geq 3$, so that the first row of $\boldsymbol{c}$ is a shifted version of the zeroth row, comprising the same elements, except for $\{c_{0,1}, c_{0,2}\} = \{1, 2\}$. Now, the rows and columns of $\boldsymbol{c}$ strictly increase, so the chains $c_1^+ \supset c_3^+ \supset c_5^+ \supset \ldots$ and $c_2^+ \supset c_4^+ \supset c_6^+ \supset \ldots$ also hold, demonstrating the second equality in (29).

Corollary 3.5 shows that $\bigcup_{k=1}^{\infty} d_k = K$. Thus the first equality in (30) follows from partition (27). To demonstrate the second equality, consider from Corollary 3.11 that $b_1 \supset b_2 \supset b_3 \supset \ldots$. As shown in Corollary 3.10 and cited in Remark 3.5, $b_{1,k-1} = b_{0,k}$, for $k \geq 2$, so that the first row of $\boldsymbol{b}$ is a shifted version of the zeroth row, comprising the same elements, except for $b_{0,1} = 1$. Now, the rows and columns of $\boldsymbol{b}$ strictly increase, so the chain $b_1^+ \supset b_2^+ \supset b_3^+ \supset \ldots$ also holds, demonstrating the second equality in (30). □

The symmetry of the quilt makes (28) easy to visualize (Figure 13). For any quilt square on or completely above the diagonal, if the selected square is size 1×1, then its western edge $a_{n,1}$ will not align with the western edge $\gamma_{n,k}$ of any rectangle below the diagonal. However, if the selected square is of size 2×2 or larger, then its western edge $a_{n,k}$, $k \geq 2$ must align with the western edge $\gamma_{n,1}$ or $\gamma_{n,2}$ of some 1×2 or 2×3 rectangle below the diagonal (vertical lines in Figure 13). Moreover, the western edge $\gamma_{n,k}, k \geq 3$ of any larger rectangle below the diagonal must also align with the western edge $\gamma_{n,1}$ or $\gamma_{n,2}$ of some 1×2 or 2×3 rectangle below the diagonal, since $\gamma_1 \supset \gamma_3 \supset \gamma_5 \supset \ldots$ and $\gamma_2 \supset \gamma_4 \supset \gamma_6 \supset \ldots$. From west to east, the arrows in Figure 13, illustrate the following relations:

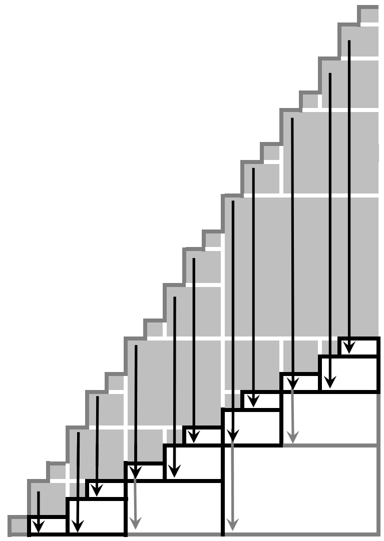


FIGURE 13. Complementarity $\{a_{n,k}\}_{n\geq 0,k\geq 2} = \gamma_1 \cup \gamma_2 = \bigcup_{k\geq 1} \gamma_k$ (28) in the quilt, from $\mathbb{Z}_{\geq 2} \setminus [\Lambda{+}1] = [K^2{+}1] \cup [K\Lambda{+}1]$ of Figure 8

$$
\begin{array}{lllll}
a_{0,2} & = \gamma_{1,1} & & & \\
a_{0,3} & & = \gamma_{1,2} & & \\
a_{1,2} & = \gamma_{2,1} & & & \\
a_{0,4} & = \gamma_{3,1} & & = \gamma_{1,3} & \\
a_{1,3} & & = \gamma_{2,2} & & \\
a_{2,2} & = \gamma_{4,1} & & & \\
a_{0,5} & & = \gamma_{3,2} & & = \gamma_{1,4} \\
a_{3,2} & = \gamma_{5,1} & & & \\
a_{1,4} & = \gamma_{6,1} & & = \gamma_{2,3} & \\
a_{2,3} & & = \gamma_{4,2} & & \\
a_{4,2} & = \gamma_{7,1} & & &
\end{array}
$$

The quilt also provides a ready view (Figure 14) of the complementarity (29). For any quilt square on or completely above the diagonal, if the selected square is size 1×1 or 2×2, then its western edge, $a_{n,1}$ or $a_{n,2}$, will not align with the western edge $c_{n,k}$ of any square below the diagonal. However, if the selected square is of size 3×3 or larger, then its western edge, $a_{n,k}$, $k \geq 3$, must align with the western edge, $c_{n,1}$ or $c_{n,2}$, of some 1×1 or 2×2 square below the diagonal (vertical lines in Figure 14). Moreover, the western edge $c_{n,k}, k \geq 3$ of any larger square on or below the diagonal must also align with the western edge, $c_{n,1}$ or $c_{n,1}$, of some 1×1 or 2×2 square below the diagonal, since $c_1 \supset c_3 \supset c_5 \supset \ldots$ and $c_2 \supset c_4 \supset c_6 \supset \ldots$. From west to east, the arrows in Figure 14, illustrate the following relations:

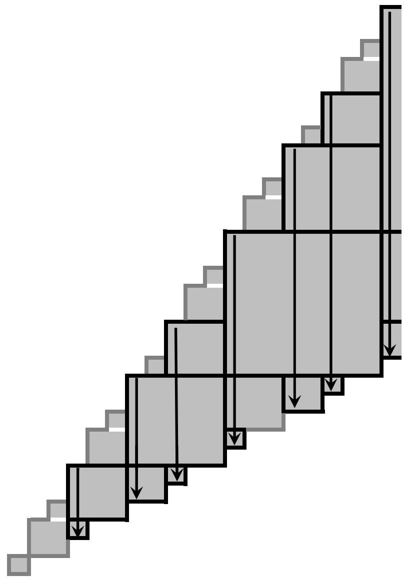


FIGURE 14. Complementarity $\{a_{n,k}\}_{n\geq 0,k\geq 3} = c_1^+ \cup c_2^+ = \bigcup_{k\geq 1} c_k^+$ (29) in the quilt, from $\mathbb{Z}_{\geq 3}\setminus[K^2{+}2]\setminus[K\Lambda{+}2] = [\Lambda K{+}2]\cup[\Lambda^2{+}2]$ of Figure 9

$$
\begin{array}{lllll}
a_{0,3} & & & & = c_{1,1} \\
a_{0,4} & & & = c_{1,2} & \\
a_{1,3} & & & & = c_{2,1} \\
a_{0,5} & & = c_{1,3} & & = c_{3,1} \\
a_{1,4} & & & = c_{2,2} & \\
a_{2,3} & & & & = c_{4,1} \\
a_{0,6} & = c_{1,4} & & = c_{3,2} &
\end{array}
$$

Finally, the quilt symmetry makes (30) plain to see (Figure 15). First observe that, for squares on or below the diagonal, the eastern edge lies in row $d_{n,k}$, whilst for squares on or above the diagonal, the eastern edge lies in row $b_{n,k}$. For any quilt square on or below the diagonal, then, if the selected square is of size 1×1, then its eastern edge $d_{n,1}$ will not align with the eastern edge $b_{n,k}$ of any square above the diagonal, but if the selected square is of size 2×2 or larger, then its eastern edge $d_{n,k}, k\geq 2$, must align with the eastern edge $b_{n,1}$ of a 1×1 square above the diagonal (vertical lines in Figure 15). Moreover, the eastern edge, $b_{n,k}, k\geq 2$, of any larger square on or above the diagonal must also align with the eastern edge $b_{n,1}$ of a 1×1 square above the diagonal, since $b_1 \supset b_2 \supset b_3 \supset \ldots$. From west to east, the arrows in Figure 15, illustrate the following relations:

$$
\begin{array}{lllll}
d_{0,2} & & & & = b_{1,1} \\
d_{0,3} & & & = b_{1,2} & = b_{2,1} \\
d_{1,2} & & & & = b_{3,1} \\
d_{0,4} & & = b_{1,3} & = b_{2,2} & = b_{4,1} \\
d_{1,3} & & & = b_{3,2} & = b_{5,1} \\
d_{2,2} & & & & = b_{6,1} \\
d_{0,5} & = b_{1,4} & = b_{2,3} & = b_{4,2} & = b_{7,1}
\end{array}
$$

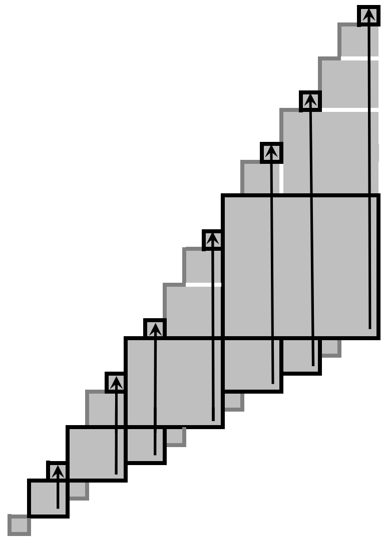


FIGURE 15. Complementarity $\{d_{n,k}\}_{n\geq 0,k\geq 2} = b_1^+ = \bigcup_{k\geq 1} b_k^+$ (30) in the quilt, from $K \setminus \{1\} \setminus [K\Lambda{+}1] = \Lambda{+}1$ of Figure 12

4. Proofs

Lemma 4.1.

$$F_{k+h} - \phi^h F_k = F_h(-\tfrac{1}{\phi})^k. \tag{31}$$

Proof. If (31) is not already known to the reader, then begin with the familiar identity

$$F_{k+1} - \phi F_k = (-\tfrac{1}{\phi})^k. \tag{32}$$

Substituting the variable in (32), write the system

$$h\begin{cases} F_{k+h} - \phi F_{k+h-1} & = (-\tfrac{1}{\phi})^{k+h-1}, \\ \vdots \\ F_{k+1} - \phi F_k & = (-\tfrac{1}{\phi})^k, \end{cases}$$

then use suitable multiples of ϕ to develop the left-hand sides into a telescoping sum:

$$h\begin{cases} \phi^{1-1}F_{k+h} - \phi^1 F_{k+h-1} & = \phi^{1-1}(-\tfrac{1}{\phi})^{k+h-1} \\ \vdots \\ \phi^{h-1}F_{k+1} - \phi^h F_k & = \phi^{h-1}(-\tfrac{1}{\phi})^k, \end{cases}$$

which collapses when added to give $F_{k+h} - \phi^h F_k = (-\frac{1}{\phi})^k \sum_{m=0}^{h-1} \phi^m (-\frac{1}{\phi})^{h-m-1}$. Finally, apply the identity

$$F_h = \sum_{m=0}^{h-1} \phi^m (-\tfrac{1}{\phi})^{h-m-1} \tag{33}$$

proving the claim. □

Proposition 4.2. *For column offset $h = 1, 2, 3, \ldots$, elements of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ satisfy, respectively, for all $k \geq 1$,*

$$\begin{aligned} \phi^h - {}^1\!/_{\phi}\ F_h - 1 &< a_{n,k+h} - a_{n,k}\phi^h < \phi^h + {}^1\!/_{\phi^2} F_h - 1, \\ 2\phi^h - {}^1\!/_{\phi^3} F_h - 2 &< b_{n,k+h} - b_{n,k}\phi^h \leq 2\phi^h + {}^1\!/_{\phi^4} F_h - 2, \\ \phi^h - {}^1\!/_{\phi^3} F_h - 1 &\leq c_{n,k+h} - c_{n,k}\phi^h \leq \phi^h + {}^1\!/_{\phi^4} F_h - 1, \\ 2\phi^h - {}^1\!/_{\phi^3} F_h - 2 &< d_{n,k+h} - d_{n,k}\phi^h < 2\phi^h + {}^1\!/_{\phi^2} F_h - 2. \end{aligned} \tag{34}$$

Tighter bounds obtain by separating the odd and even cases, thus for k odd,

$$\begin{aligned} \phi^h - {}^1\!/_{\phi^k}\ F_h - 1 &< a_{n,k+h} - a_{n,k}\phi^h \leq \phi^h - {}^1\!/_{\phi^{k+2}} F_h - 1, \\ 2\phi^h - {}^1\!/_{\phi^{k+2}} F_h - 2 &< b_{n,k+h} - b_{n,k}\phi^h \leq 2\phi^h + {}^1\!/_{\phi^{k+3}} F_h - 2, \\ \phi^h - {}^1\!/_{\phi^{k+2}} F_h - 1 &\leq c_{n,k+h} - c_{n,k}\phi^h < \phi^h - 1, \\ 2\phi^h + {}^1\!/_{\phi^{k+3}} F_h - 2 &\leq d_{n,k+h} - d_{n,k}\phi^h < 2\phi^h + {}^1\!/_{\phi^{k+1}} F_h - 2; \end{aligned} \tag{35}$$

while for k even,

$$\begin{aligned} \phi^h + {}^1\!/_{\phi^{k+2}} F_h - 1 &\leq a_{n,k+h} - a_{n,k}\phi^h < \phi^h + {}^1\!/_{\phi^k}\ F_h - 1, \\ 2\phi^h - {}^1\!/_{\phi^{k+3}} F_h - 2 &\leq b_{n,k+h} - b_{n,k}\phi^h < 2\phi^h + {}^1\!/_{\phi^{k+2}} F_h - 2, \\ \phi^h - 1 &< c_{n,k+h} - c_{n,k}\phi^h \leq \phi^h + {}^1\!/_{\phi^{k+2}} F_h - 1, \\ 2\phi^h - {}^1\!/_{\phi^{k+1}} F_h - 2 &< d_{n,k+h} - d_{n,k}\phi^h \leq 2\phi^h - {}^1\!/_{\phi^{k+3}} F_h - 2. \end{aligned} \tag{36}$$

Proof of inequalities (34), (35), and (36) in $\boldsymbol{a}$. Begin by rewriting

$$0 \leq m\phi - \lfloor m\phi \rfloor < 1 \text{ as} \tag{37}$$

$$0 \leq m \ - \frac{\lfloor m\phi \rfloor}{\phi} < \frac{1}{\phi}, \tag{38}$$

and proceed to manipulate (38) into the desired result. Write out the quantity $a_{n,k+h} - a_{n,k}\phi^h$ in (34) using the cohort-based formula (1) and substitute (31) to obtain $a_{n,k+h} - a_{n,k}\phi^h = F_h(-\frac{1}{\phi})^{k+2} + \lfloor n\phi \rfloor F_h(-\frac{1}{\phi})^{k+1} + nF_h(-\frac{1}{\phi})^k + \phi^h - 1$. In the latter expression, observe that the coefficient of the $\lfloor n\phi \rfloor$ term is $-\frac{1}{\phi}$ times the coefficient of the n term, just as in the expression (38). Now, for $k = 1, 3, 5, \ldots$, (38) gives $\phi^h - 1 - \frac{F_h}{\phi} \leq \phi^h - 1 - F_h(\frac{1}{\phi})^k = \phi^h - 1 - F_h((\frac{1}{\phi})^{k+1} + (\frac{1}{\phi})^{k+2}) < a_{n,k+h} - a_{n,k}\phi^h \leq \phi^h - 1 - F_h(\frac{1}{\phi})^{k+2} < \phi^h - 1$, or (35). For $k = 2, 4, 6, \ldots$, (38) gives $\phi^h - 1 < \phi^h - 1 + F_h(\frac{1}{\phi})^{k+2} \leq a_{n,k+h} - a_{n,k}\phi^h < \phi^h - 1 + F_h((\frac{1}{\phi})^{k+1} + (\frac{1}{\phi})^{k+2}) = \phi^h - 1 + F_h(\frac{1}{\phi})^k \leq \phi^h - 1 + \frac{F_h}{\phi^2}$, or (36). Combining the cases for k odd and k even, obtain looser bounds (34) valid for all $k \geq 1$.

Proofs for $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are similar. ∎

Corollary 4.3. *For column offset $h = 1, 2, 3, \ldots$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ satisfy*

$$\begin{aligned} 1 - {}^1\!/_{\phi^h} - {}^{F_h}\!/_{\phi^{h+1}} &< {}^{a_{n,k+h}}\!/_{\phi^h} - a_{n,k} < 1 - {}^1\!/_{\phi^h} + {}^{F_h}\!/_{\phi^{h+2}}, \\ 2 - {}^2\!/_{\phi^h} - {}^{F_h}\!/_{\phi^{h+3}} &< {}^{b_{n,k+h}}\!/_{\phi^h} - b_{n,k} \leq 2 - {}^2\!/_{\phi^h} + {}^{F_h}\!/_{\phi^{h+4}}, \\ 1 - {}^1\!/_{\phi^h} - {}^{F_h}\!/_{\phi^{h+3}} &\leq {}^{c_{n,k+h}}\!/_{\phi^h} - c_{n,k} \leq 1 - {}^1\!/_{\phi^h} + {}^{F_h}\!/_{\phi^{h+4}}, \\ 2 - {}^2\!/_{\phi^h} - {}^{F_h}\!/_{\phi^{h+3}} &\leq {}^{d_{n,k+h}}\!/_{\phi^h} - d_{n,k} < 2 - {}^2\!/_{\phi^h} + {}^{F_h}\!/_{\phi^{h+2}}. \end{aligned}$$

Proposition 4.4. *For column offset $h = 1, 2, 3, \ldots$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ satisfy*

$$\begin{aligned} F_{h+1} - 1 &< a_{n,k+h} - a_{n,k}\phi^h < F_{h+2} - 1, \\ F_{h+3} - 2 &< b_{n,k+h} - b_{n,k}\phi^h < F_{h+3} - 2 + F_h(\tfrac{1}{\phi})^2, \\ F_{h+1} - 1 + F_h(\tfrac{1}{\phi})^2 &< c_{n,k+h} - c_{n,k}\phi^h < F_{h+2} - 1 - F_h(\tfrac{1}{\phi})^3, \\ F_{h+3} - 2 &< d_{n,k+h} - d_{n,k}\phi^h < F_{h+3} - 2 + F_h(\tfrac{1}{\phi}). \end{aligned} \tag{39}$$

Proof of inequalities (39) in $\boldsymbol{a}$. As in the previous proof, begin by rewriting (37) as (38), and proceed to manipulate (38) into the desired result. Write out the quantity $a_{n,k+h} - a_{n,k}\phi^h$ in(39) using the cohort-based formula (1) and substitute (31) to obtain $a_{k+h,n} - a_{k,n}\phi^h = F_h(-\frac{1}{\phi})^{k+2} + \lfloor n\phi \rfloor F_h(-\frac{1}{\phi})^{k+1} + nF_h(-\frac{1}{\phi})^k + \phi^h - 1$. For $k = 1, 3, 5, \ldots$ odd, (38) gives $\phi^h - 1 - \frac{F_h}{\phi} \leq \phi^h - 1 - F_h(\frac{1}{\phi})^k = \phi^h - 1 - F_h((\frac{1}{\phi})^{k+1} + (\frac{1}{\phi})^{k+2}) < a_{n,k+h} - a_{n,k}\phi^h \leq \phi^h - 1 - F_h(\frac{1}{\phi})^{k+2} < \phi^h - 1$. For $k = 2, 4, 6, \ldots$, (38) gives $\phi^h - 1 < \phi^h - 1 + F_h(\frac{1}{\phi})^{k+2} \leq a_{n,k+h} - a_{n,k}\phi^h < \phi^h - 1 + F_h((\frac{1}{\phi})^{k+1} + (\frac{1}{\phi})^{k+2}) = \phi^h - 1 + F_h(\frac{1}{\phi})^k \leq \phi^h - 1 + \frac{F_h}{\phi^2}$. Combining the cases for k odd and k even, obtain bounds on row recursion in $\boldsymbol{a}$ that reduce to the desired bounds (39), and similar bounds for $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$:

$$\begin{aligned} \phi^h - 1 - \tfrac{F_h}{\phi} &< a_{n,k+h} - a_{n,k}\phi^h < \phi^h - 1 + \tfrac{F_h}{\phi^2}, \\ 2\phi^h - 2 - \tfrac{F_h}{\phi^3} &< b_{n,k+h} - b_{n,k}\phi^h < 2\phi^h - 2 + \tfrac{F_h}{\phi^4}, \\ \phi^h - 1 - \tfrac{F_h}{\phi^3} &< c_{n,k+h} - c_{n,k}\phi^h < \phi^h - 1 + \tfrac{F_h}{\phi^4}, \\ 2\phi^h - 2 - \tfrac{F_h}{\phi^3} &< d_{n,k+h} - d_{n,k}\phi^h < 2\phi^h - 2 + \tfrac{F_h}{\phi^2}. \end{aligned}$$

Proofs for $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are similar. □

Proof. (of Corollary 3.4) Cohort-based formulas (1) and (4) show that for $n = 0, 1, 2, \ldots$ and $k = 1, 2, 3 \ldots$, $a_{n,k}$ and $d_{n,k}$ are integers, so it suffices to prove $0 \leq a_{n,k}\phi - d_{n,k} < 1$, and since they are positive integers, this becomes, without loss of generality

$$0 < a_{n,k}\phi - d_{n,k} < 1. \tag{40}$$

Similarly to the proofs of Propositions 4.2 and 4.4, to show (40) proceed by manipulating (38) into (40), considering separate cases of k odd and k even. First consider $k \geq 1$ odd. Then, $(-\frac{1}{\phi})^k < 0$, $(-\frac{1}{\phi})^{k+1} > 0$, and $(-\frac{1}{\phi})^{k+2} < 0$. Take $m = n$ in (38) and multiply by $(-\frac{1}{\phi})^k < 0$ to obtain $0 \geq (-\frac{1}{\phi})^k n + (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor > -(-\frac{1}{\phi})^{k+1}$. Add $(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^2$ to obtain $(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^2 \geq (-\frac{1}{\phi})^{k+2} + (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor + (-\frac{1}{\phi})^k n - (-\frac{1}{\phi})^2 > (-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^{k+1} - (-\frac{1}{\phi})^2$. Multiply by -1 to obtain $(-\frac{1}{\phi})^2 - (-\frac{1}{\phi})^{k+2} \leq -(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor - (-\frac{1}{\phi})^k n + (-\frac{1}{\phi})^2 < (-\frac{1}{\phi})^2 + (-\frac{1}{\phi})^{k+1} - (-\frac{1}{\phi})^{k+2}$, which for $k \geq 1$ odd satisfies $0 < (\frac{1}{\phi})^2 < (\frac{1}{\phi})^2 + (\frac{1}{\phi})^{k+2} \leq -(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor - (-\frac{1}{\phi})^k n + (-\frac{1}{\phi})^2 < (\frac{1}{\phi})^2 + (\frac{1}{\phi})^{k+1} + (\frac{1}{\phi})^{k+2} \leq (\frac{1}{\phi})^2 + (\frac{1}{\phi})^2 + (\frac{1}{\phi})^3 = 1$, or simply $0 < -(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor - (-\frac{1}{\phi})^k n + (-\frac{1}{\phi})^2 < 1$. Finally apply the identity (32) and the particular case $(-\frac{1}{\phi})^2 = 2 - \phi$ to rewrite the expression as $0 < -(F_{k+3} - \phi F_{k+2}) - \lfloor n\phi \rfloor (F_{k+2} - \phi F_{k+1}) - n(F_{k+1} - \phi F_k) + (2 - \phi) < 1$. After rearranging and substituting in (1) and (4) this becomes (40), as desired.

For $k \geq 1$ even, the argument is similar except that $(-\frac{1}{\phi})^k > 0$, $(-\frac{1}{\phi})^{k+1} < 0$, and $(-\frac{1}{\phi})^{k+2} > 0$. Take $m = n$ in (38) and multiply by $(-\frac{1}{\phi})^k > 0$ to obtain

$0 \leq (-\frac{1}{\phi})^k n + (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor < -(-\frac{1}{\phi})^{k+1}$. Add $(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^2$ to obtain $(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^2 \leq (-\frac{1}{\phi})^{k+2} + (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor + (-\frac{1}{\phi})^k n - (-\frac{1}{\phi})^2 < (-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^{k+1} - (-\frac{1}{\phi})^2$. Multiply by -1 to obtain $(-\frac{1}{\phi})^2 + (-\frac{1}{\phi})^{k+1} - (-\frac{1}{\phi})^{k+2} < -(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor - (-\frac{1}{\phi})^k n + (-\frac{1}{\phi})^2 \leq (-\frac{1}{\phi})^2 - (-\frac{1}{\phi})^{k+2}$, which for $k \geq 1$ even satisfies $0 = (\frac{1}{\phi})^2 - (\frac{1}{\phi})^3 - (\frac{1}{\phi})^4 \leq (\frac{1}{\phi})^2 - (\frac{1}{\phi})^{k+1} - (\frac{1}{\phi})^{k+2} < -(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor - (-\frac{1}{\phi})^k n + (-\frac{1}{\phi})^2 \leq (\frac{1}{\phi})^2 - (\frac{1}{\phi})^{k+2} < (\frac{1}{\phi})^2 < 1$, or simply $0 < -(-\frac{1}{\phi})^{k+2} - (-\frac{1}{\phi})^{k+1} \lfloor n\phi \rfloor - (-\frac{1}{\phi})^k n + (-\frac{1}{\phi})^2 < 1$. The latter expression, as in the odd case, is equivalent to (40). □

Lemma 4.5 (Cohort index of array entries in the branch and clade quartets)**.** *For the equalities established above, quantify the values for $k \geq 1$:*

$$(41) \quad F^{-1}(\mathsf{F}_{n,k}) = F^{-1}(\text{⊑}_{n,k}) = F^{-1}(w_{n,k}) = F^{-1}(a_{n,k}) = F^{-1}(n) + k+1, n \geq 0.$$

$$(42) \quad F^{-1}(\text{ꟻ}_{n,k}) = F^{-1}(\text{⊒}_{n,k}) = F^{-1}(\mathfrak{w}_{n,k}) = F^{-1}(\text{ɒ}_{n,k}) = \begin{cases} 2k, & n = 0; \\ F^{-1}(n) + 2k - 1, & n \geq 1. \end{cases}$$

Proof. In Part 2 of this paper [6]. □

Proof of Proposition 3.8 for (F)*:* Since $\mathsf{F}_{n,k-1} \geq 1, \forall n \geq 0, k \geq 2$, write as a single case and take it through successive simplifications,

$$\begin{aligned}
\mathsf{F}_{n,k} &= \text{ꟻ}_{\mathsf{F}_{n,k-1},1} \\
n + F_{F^{-1}(n)+k+1} &= n + F_{F^{-1}(n)+k} - 2F_{F^{-1}(n+F_{F^{-1}(n)+k})} + F_{F^{-1}(n+F_{F^{-1}(n)+k})+2} \\
F_{F^{-1}(n)+k-1} &= -2F_{F^{-1}(n+F_{F^{-1}(n)+k})} + F_{F^{-1}(n+F_{F^{-1}(n)+k})+2} \\
F_{F^{-1}(n)+k-1} &= F_{F^{-1}(n+F_{F^{-1}(n)+k})-1} \\
F^{-1}(n) + k &= F^{-1}(n + F_{F^{-1}(n)+k}) \\
F^{-1}(n) + (k-1) + 1 &= F^{-1}(n + F_{F^{-1}(n)+(k-1)+1}) \\
F^{-1}(n) + (k-1) + 1 &= F^{-1}(\mathsf{F}_{n,k-1}),
\end{aligned}$$

where (41) shows the latter for for $k - 1 \geq 1$. □

Proof of Proposition 3.8 for (ꟻ)*:* Case $n = 0$: For $k \geq 2$ taking the claim through successive simplifications,

$$\begin{aligned}
\text{ꟻ}_{0,k} &= \mathsf{F}_{\text{ꟻ}_{0,k-1},1} \\
F_{2k+1} - 1 &= F_{2k-1} - 1 + F_{F^{-1}(F_{2k-1}-1)+2} \\
F_{2k} &= F_{F^{-1}(F_{2k-1}-1)+2} \\
2k - 2 &= F^{-1}(F_{2k-1} - 1) \\
2(k-1) &= F^{-1}(F_{2(k-1)+1} - 1) \\
2(k-1) &= F^{-1}(\text{ꟻ}_{0,k-1}),
\end{aligned}$$

where (42) shows the latter for $n = 0, k - 1 \geq 1$.

Case $n \geq 1$: For $n \geq 1, k \geq 2$ taking the claim through successive simplifications,

$$\begin{aligned}
\text{ꟻ}_{n,k} &= \mathsf{F}_{\text{ꟻ}_{n,k-1},1} \\
n + F_{F^{-1}(n)+2k} - 2F_{F^{-1}(n)} &= n + F_{F^{-1}(n)+2(k-1)} - 2F_{F^{-1}(n)} \\
&\quad + F_{F^{-1}(n+F_{F^{-1}(n)+2(k-1)}-2F_{F^{-1}(n)})+2} \\
F_{F^{-1}(n)+2(k-1)+1} &= F_{F^{-1}(n+F_{F^{-1}(n)+2(k-1)}-2F_{F^{-1}(n)})+2} \\
F^{-1}(n) + 2(k-1) - 1 &= F^{-1}(n + F_{F^{-1}(n)+2(k-1)} - 2F_{F^{-1}(n)}) \\
F^{-1}(n) + 2(k-1) - 1 &= F^{-1}(\text{ꟻ}_{n,k-1}),
\end{aligned}$$

where (42) shows the latter for for $n \geq 1, k-1 \geq 1$. □

Proof of Proposition 3.8 for (L)*:* Since $\mathsf{L}_{n,k-1} \geq 1, \forall n \geq 0, k \geq 2$, write as a single case and take it through successive simplifications,

$$\begin{aligned}
\mathsf{L}_{n,k} &= \text{⅃}_{\mathsf{L}_{n,k-1},1} \\
n + F_{F^{-1}(n)+k+2} - F_{F^{-1}(n)+2} &= n + F_{F^{-1}(n)+k+1} - F_{F^{-1}(n)+2} \\
&\quad + F_{F^{-1}(n+F_{F^{-1}(n)+k+1}-F_{F^{-1}(n)+2})} \\
F_{F^{-1}(n)+k} &= F_{F^{-1}(n+F_{F^{-1}(n)+k+1}-F_{F^{-1}(n)+2})} \\
F^{-1}(n) + k &= F^{-1}(n + F_{F^{-1}(n)+k+1} - F_{F^{-1}(n)+2}) \\
F^{-1}(n) + (k-1) + 1 &= F^{-1}(n + F_{F^{-1}(n)+(k-1)+2} - F_{F^{-1}(n)+2}) \\
F^{-1}(n) + (k-1) + 1 &= F^{-1}(\mathsf{L}_{n,k-1}),
\end{aligned}$$

where (41) showed the latter for $k-1 \geq 1$. □

Proof of Proposition 3.8 for (⅃)*:* Case $n = 0$: For $k \geq 2$, taking the claim through successive simplifications,

$$\begin{aligned}
\text{⅃}_{0,k} &= \mathsf{L}_{\text{⅃}_{0,k-1},1} \\
F_{2k} &= F_{2(k-1)} + F_{F^{-1}(F_{2(k-1)})+1} \\
F_{2k-1} &= F_{2(k-1)+1}
\end{aligned}$$

which is trivial.

Case $n \geq 1$: For $n \geq 1, k \geq 2$, taking the claim through successive simplifications,

$$\begin{aligned}
\text{⅃}_{n,k} &= \mathsf{L}_{\text{⅃}_{n,k-1},1} \\
n + F_{F^{-1}(n)+2k-1} - F_{F^{-1}(n)-1} &= n + F_{F^{-1}(n)+2(k-1)-1} - F_{F^{-1}(n)-1} \\
&\quad + F_{F^{-1}(n+F_{F^{-1}(n)+2(k-1)-1}-F_{F^{-1}(n)-1})+1} \\
F_{F^{-1}(n)+2k-2} &= F_{F^{-1}(n+F_{F^{-1}(n)+2(k-1)-1}-F_{F^{-1}(n)-1})+1} \\
F^{-1}(n) + 2k - 2 &= F^{-1}(n + F_{F^{-1}(n)+2(k-1)-1} - F_{F^{-1}(n)-1}) + 1 \\
F^{-1}(n) + 2(k-1) - 1 &= F^{-1}(\text{⅃}_{n,k-1})
\end{aligned}$$

where (42) showed the latter for $n \geq 1, k-1 \geq 1$. □

Proof of Proposition 3.8 for (w)*:* Since $w_{n,k-1} \geq 1, \forall n \geq 0, k \geq 2$, write as a single case $\mathfrak{w}_{w_{n,k-1},1} = \kappa(F_k\kappa(n+1) + F_{k-1}n + 1) - 1$. Now, substitute $m = w_{n,k-1} + 1 = F_k\kappa(n+1) + F_{k-1}n + 1$ into (37) to obtain

$$0 \leq \phi(F_k\kappa(n+1) + F_{k-1}n + 1) - \kappa(F_k\kappa(n+1) + F_{k-1}n + 1) < 1.$$

Next, using (32), write this as

$$0 \leq \phi(F_{k+1} - (-\tfrac{1}{\phi})^k)\kappa(n+1) + (F_k - (-\tfrac{1}{\phi})^{k-1})n + \phi \\ - \kappa(F_k\kappa(n+1) + F_{k-1}n + 1) < 1,$$

or

$$\kappa(n+1)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} - \phi \\ \leq (F_{k+1}\kappa(n+1) + F_k n) - \kappa(F_k\kappa(n+1) + F_{k-1}n + 1) \\ < 1 + \kappa(n+1)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} - \phi.$$

Simplify the lower bound $\kappa(n+1)(-\frac{1}{\phi})^k + n(-\frac{1}{\phi})^{k-1} - \phi = (-\frac{1}{\phi})^{k-1}(-\frac{1}{\phi}\kappa(n+1) + n) - \phi$, considering that minima of $n - \frac{1}{\phi}\kappa(n+1)$ occur at $n = F_{2m+1} - 1$, so that

$$\begin{aligned}
&\min_{n,k} \kappa(n+1)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} - \phi \\
=&\min_{m,k}(-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}\kappa(F_{2m+1}) + F_{2m+1} - 1) - \phi \\
=&\min_{m,k}(-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}F_{2m+2} + F_{2m+1} - 1) - \phi \\
=&\min_{m,k}(-\tfrac{1}{\phi})^{k-1}((\tfrac{1}{\phi})^{2m+2} - 1) - \phi \\
=&\min_{k} \lim_{m\to\infty}(-\tfrac{1}{\phi})^{k-1}((\tfrac{1}{\phi})^{2m+2} - 1) - \phi \\
=&\min_{k\geq 2} -(-\tfrac{1}{\phi})^{k-1} - \phi \\
=&-(\tfrac{1}{\phi})^2 - \phi \\
=&-2.
\end{aligned}$$

Simplify the upper bound $1+\kappa(n+1)(-\frac{1}{\phi})^k + n(-\frac{1}{\phi})^{k-1} - \phi = 1 + (-\frac{1}{\phi})^{k-1}(-\frac{1}{\phi}\kappa(n+1) + n) - \phi$, considering that the minima of $n - \frac{1}{\phi}\kappa(n+1)$ occur at $n = F_{2m+1} - 1$ and the maxima at $n = F_{2m+2} - 1$, so that either

$$\begin{aligned}
&\max_{n,k} 1 + \kappa(n+1)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} - \phi \\
=&\max_{m,k} 1 + (-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}\kappa(F_{2m+1}) + F_{2m+1} - 1) - \phi \\
=&\max_{m,k} 1 + (-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}F_{2m+2} + F_{2m+1} - 1) - \phi \\
=&\max_{m,k} 1 + (-\tfrac{1}{\phi})^{k-1}((\tfrac{1}{\phi})^{2m+2} - 1) - \phi \\
=&\max_{k} \lim_{m\to\infty} 1 + (-\tfrac{1}{\phi})^{k-1}((\tfrac{1}{\phi})^{2m+2} - 1) - \phi \\
=&\max_{k\geq 2} 1 - (-\tfrac{1}{\phi})^{k-1} - \phi \\
=&\max_{k\geq 2} 1 + \tfrac{1}{\phi} - \phi \\
=&0,
\end{aligned}$$

or

$$
\begin{aligned}
&\max_{n,k} 1+\kappa(n+1)(-\tfrac{1}{\phi})^{k}+n(-\tfrac{1}{\phi})^{k-1}-\phi\\
=&\max_{m,k} 1+(-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}\kappa(F_{2m+2})+F_{2m+2}-1)-\phi\\
=&\max_{m,k} 1+(-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}(F_{2m+3}-1)+F_{2m+2}-1)-\phi\\
=&\max_{m,k} 1+(-\tfrac{1}{\phi})^{k-1}(-(\tfrac{1}{\phi})^{2m+3}-1+\tfrac{1}{\phi})-\phi\\
=&\max_{k}\lim_{m\to\infty} 1+(-\tfrac{1}{\phi})^{k-1}(-(\tfrac{1}{\phi})^{2m+3}-1+\tfrac{1}{\phi})-\phi\\
=&\max_{k} 1+(-\tfrac{1}{\phi})^{k-1}(-1+\tfrac{1}{\phi})-\phi\\
=&\max_{k} 1+(\tfrac{1}{\phi})^{2}(-1+\tfrac{1}{\phi})-\phi\\
=&1-\phi-(\tfrac{1}{\phi})^{2}+(\tfrac{1}{\phi})^{3}\\
<&0.
\end{aligned}
$$

Continue to manipulate the formula until the desired difference is bounded:

$$
\begin{aligned}
-2 <& F_{k+1}\kappa(n+1)+F_kn - \kappa(F_k\kappa(n+1)+F_{k-1}n+1) &< 0,\\
-2 <& w_{n,k} \qquad\qquad -(\mathfrak{w}_{w_{n,k-1},1}+1) &< 0,\\
-1 <& w_{n,k} \qquad\qquad -\mathfrak{w}_{w_{n,k-1},1} &< 1.
\end{aligned}
$$

The two positive integer quantities $w_{n,k}$ and $\mathfrak{w}_{w_{n,k-1},1}$ have a difference of less than one thus proving the formula. □

Proof of Proposition 3.8 for ($\mathfrak{w}$)*:* Write

$$
w_{\mathfrak{w}_{n,k-1},1}=\begin{cases}\kappa(F_{2k-1})+F_{2k-1}-1, & n=0;\\ \kappa(F_{2k-3}\kappa(n+1)+F_{2k-4}n)+F_{2k-3}\kappa(n+1)+F_{2k-4}n-1, & n\geq 1.\end{cases}
$$

Case $n=0$: Simplify to $w_{\mathfrak{w}_{0,k-1},1}=\kappa(F_{2k-1})+F_{2k-1}-1=F_{2k}+F_{2k-1}-1=F_{2k+1}-1=\mathfrak{w}_{0,k}$, as claimed.

Case $n\geq 1$: Substitute $m=\mathfrak{w}_{n,k-1}+1=F_{2k-3}\kappa(n+1)+F_{2k-4}n$ into (37) to obtain

$$
0\leq \phi(F_{2k-3}\kappa(n+1)+F_{2k-4}n)-\kappa(F_{2k-3}\kappa(n+1)+F_{2k-4}n)<1.
$$

Next, using (32), write this as

$$
\begin{aligned}
0\leq (F_{2k-2}-(-\tfrac{1}{\phi})^{2k-3})\kappa(n+1)+(F_{2k-3}-(-\tfrac{1}{\phi})^{2k-4})n&\\
-\kappa(F_{2k-3}\kappa(n+1)+F_{2k-4}n)&<1,
\end{aligned}
$$

or

$$
\begin{aligned}
&(-\tfrac{1}{\phi})^{2k-3}\kappa(n+1)+(-\tfrac{1}{\phi})^{2k-4}n\\
&\quad\leq F_{2k-2}\kappa(n+1)+F_{2k-3}n-\kappa(F_{2k-3}\kappa(n+1)+F_{2k-4}n-1)\\
&\qquad\qquad< 1+(-\tfrac{1}{\phi})^{2k-3}\kappa(n+1)+(-\tfrac{1}{\phi})^{2k-4}n.
\end{aligned}
$$

Simplify the lower bound $(-\frac{1}{\phi})^{2k-3}\kappa(n+1)+(-\frac{1}{\phi})^{2k-4}n\ (\frac{1}{\phi})^{2k-4}(-\frac{1}{\phi}\kappa(n+1)+n)$, considering that the minima of $n-\frac{1}{\phi}\kappa(n+1)$ occur at $n=F_{2m+1}-1$, so that

$$\begin{aligned}
&\min_{n,k}(-\tfrac{1}{\phi})^{2k-3}\kappa(n+1)+(-\tfrac{1}{\phi})^{2k-4}n\\
=&\min_{m,k}(\tfrac{1}{\phi})^{2k-4}(-\tfrac{1}{\phi}\kappa(F_{2m+1})+F_{2m+1}-1)\\
=&\min_{m,k}(\tfrac{1}{\phi})^{2k-4}(-\tfrac{1}{\phi}F_{2m+2}+F_{2m+1}-1)\\
=&\min_{m,k}(\tfrac{1}{\phi})^{2k-4}((\tfrac{1}{\phi})^{2m+2}-1)\\
=&\min_{k\geq 2}\lim_{m\to\infty}(\tfrac{1}{\phi})^{2k-4}((\tfrac{1}{\phi})^{2m+2}-1)\\
=&\min_{k\geq 2}-(\tfrac{1}{\phi})^{2k-4}\\
=&-1,
\end{aligned}$$

where one of the steps uses (32).

Simplify the upper bound $1+(-\frac{1}{\phi})^{2k-3}\kappa(n+1)+(-\frac{1}{\phi})^{2k-4}n = 1+(\frac{1}{\phi})^{2k-4}(-\frac{1}{\phi}\kappa(n+1)+n)$, considering that the maxima of $n-\frac{1}{\phi}\kappa(n+1)$ occur at $n=F_{2m+2}-1$, so that

$$\begin{aligned}
&\max_{n,k}1+(-\tfrac{1}{\phi})^{2k-3}\kappa(n+1)+(-\tfrac{1}{\phi})^{2k-4}n\\
=&\max_{m,k}1+(\tfrac{1}{\phi})^{2k-4}(-\tfrac{1}{\phi}\kappa(F_{2m+2})+F_{2m+2}-1)\\
=&\max_{m,k}1+(\tfrac{1}{\phi})^{2k-4}(-\tfrac{1}{\phi}(F_{2m+3}-1)+F_{2m+2}-1)\\
=&\max_{m,k}1+(\tfrac{1}{\phi})^{2k-4}(\tfrac{1}{\phi}-1-\tfrac{1}{\phi}F_{2m+3}+F_{2m+2})\\
=&\max_{m,k}1+(\tfrac{1}{\phi})^{2k-4}(\tfrac{1}{\phi}-1+(-\tfrac{1}{\phi})^{2m+3})\\
=&\max_{k}\lim_{m\to\infty}1+(\tfrac{1}{\phi})^{2k-4}(\tfrac{1}{\phi}-1+(-\tfrac{1}{\phi})^{2m+3})\\
=&\lim_{k\to\infty}1+(\tfrac{1}{\phi})^{2k-4}(\tfrac{1}{\phi}-1)\\
=&1,
\end{aligned}$$

Continue to manipulate the formula until the desired difference is bounded:

$$\begin{aligned}
-1<&F_{2k-2}\kappa(n+1)+F_{2k-3}n-\kappa(F_{2k-3}\kappa(n+1)+F_{2k-4}n) &&<1+\phi,\\
-1<&F_{2k-1}\kappa(n+1)+F_{2k-2}n-[F_{2k-3}\kappa(n+1)+F_{2k-4}n\\
&\qquad +\kappa(F_{2k-3}\kappa(n+1)+F_{2k-4}n)] &&<1,\\
-1<&\mathfrak{w}_{n,k}+1 \quad -(w_{\mathfrak{w}_{n,k-1},1}+1) &&<1,\\
-1<&\mathfrak{w}_{n,k} \quad -w_{\mathfrak{w}_{n,k-1},1} &&<1.
\end{aligned}$$

The two positive integer quantities $\mathfrak{w}_{n,k}$ and $w_{\mathfrak{w}_{n,k-1},1}$ have a difference of less than one thus proving the formula. □

Proof of Proposition 3.8 for (a)*:* Substitute $m=a_{n,k-1}=F_k\kappa(n)+F_{k-1}n+F_{k+1}-1$ into (37) to obtain

$$0\leq\phi(F_k\kappa(n)+F_{k-1}n+F_{k+1}-1)-\kappa(F_k\kappa(n)+F_{k-1}n+F_{k+1}-1)<1.$$

Next, using (32), write this as

$$0 \le (F_{k+1} - (-\tfrac{1}{\phi})^k)\kappa(n) + (F_k - (-\tfrac{1}{\phi})^{k-1})n + (F_{k+2} - (-\tfrac{1}{\phi})^{k+1}) - \phi \\ - \kappa(F_k\kappa(n) + F_{k-1}n + F_{k+1} - 1) < 1,$$

or

$$\phi + \kappa(n)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} + (-\tfrac{1}{\phi})^{k+1} \\ \le F_{k+1}\kappa(n) + F_k n + F_{k+2} - \kappa(F_k\kappa(n) + F_{k-1}n + F_{k+1} - 1) \\ < \phi + 1 + \kappa(n)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} + (-\tfrac{1}{\phi})^{k+1}.$$

Simplify the lower bound $\phi + \kappa(n)(-\frac{1}{\phi})^k + n(-\frac{1}{\phi})^{k-1} + (-\frac{1}{\phi})^{k+1} = \phi + (-\frac{1}{\phi})^{k-1}$ $(-\frac{1}{\phi}\kappa(n) + n + (\frac{1}{\phi})^2)$, considering that the maxima of $n - \frac{1}{\phi}\kappa(n)$ occur at $n = F_{2m}$, so that

$$\begin{aligned}
&\min_{n,k} \phi + \kappa(n)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} + (-\tfrac{1}{\phi})^{k+1} \\
=&\min_{m,k} \phi + (-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}\kappa(F_{2m}) + F_{2m} + (\tfrac{1}{\phi})^2) \\
=&\min_{m,k} \phi + (-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}(F_{2m+1} - 1) + F_{2m} + (\tfrac{1}{\phi})^2) \\
=&\min_{m,k} \phi + (-\tfrac{1}{\phi})^{k-1}(1 - \tfrac{1}{\phi}F_{2m+1} + F_{2m}) \\
=&\min_{m,k} \phi + (-\tfrac{1}{\phi})^{k-1}(1 + (-\tfrac{1}{\phi})^{2m+1}) \\
=&\min_{k\ge 2} \lim_{m\to\infty} \phi + (-\tfrac{1}{\phi})^{k-1}(1 + (-\tfrac{1}{\phi})^{2m+1}) \\
=&\min_{k\ge 2} \phi + (-\tfrac{1}{\phi})^{k-1} \\
=&\phi - \tfrac{1}{\phi} \\
=&1.
\end{aligned}$$

Simplify the upper bound $\phi + 1 + \kappa(n)(-\frac{1}{\phi})^k + n(-\frac{1}{\phi})^{k-1} + (-\frac{1}{\phi})^{k+1} = \phi + 1 +$ $(-\frac{1}{\phi})^{k-1}(-\frac{1}{\phi}\kappa(n) + n + (\frac{1}{\phi})^2)$, considering that the minimum of $n - \frac{1}{\phi}\kappa(n)$ occurs at 0 and the maxima occur at $n = F_{2m}$, so that either

$$\begin{aligned}
&\max_{n,k} \phi + 1 + \kappa(n)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} + (-\tfrac{1}{\phi})^{k+1} \\
=&\max_{k\ge 2} \phi + 1 + (-\tfrac{1}{\phi})^{k+1} \\
=&\phi + 1 + (\tfrac{1}{\phi})^4,
\end{aligned}$$

or else

$$\begin{aligned}
&\max_{n,k} \phi + 1 + \kappa(n)(-\tfrac{1}{\phi})^k + n(-\tfrac{1}{\phi})^{k-1} + (-\tfrac{1}{\phi})^{k+1} \\
=&\min_{m,k} \phi + 1 + (-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}\kappa(F_{2m}) + F_{2m} + (\tfrac{1}{\phi})^2) \\
=&\min_{m,k} \phi + 1 + (-\tfrac{1}{\phi})^{k-1}(-\tfrac{1}{\phi}(F_{2m+1} - 1) + F_{2m} + (\tfrac{1}{\phi})^2) \\
=&\min_{m,k} \phi + 1 + (-\tfrac{1}{\phi})^{k-1}(1 - \tfrac{1}{\phi}F_{2m+1} + F_{2m}) \\
=&\min_{m,k} \phi + 1 + (-\tfrac{1}{\phi})^{k-1}(1 + (-\tfrac{1}{\phi})^{2m+1}) \\
=&\min_{k\geq 2} \lim_{m\to\infty} \phi + 1 + (-\tfrac{1}{\phi})^{k-1}(1 + (-\tfrac{1}{\phi})^{2m+1}) \\
=&\min_{k\geq 2} \phi + 1 + (-\tfrac{1}{\phi})^{k-1} \\
=&\phi + 1 + (\tfrac{1}{\phi})^2 \\
=&3.
\end{aligned}$$

Continue to manipulate the formula until the desired difference is bounded:

$$\begin{aligned}
1 <& F_{k+1}\kappa(n) + F_k n + F_{k+2} && -\kappa(F_k\kappa(n) + F_{k-1}n + F_{k+1} - 1) && < 3. \\
-1 <& F_{k+1}\kappa(n) + F_k n + F_{k+2} - 1 && -(\kappa(F_k\kappa(n) + F_{k-1}n + F_{k+1} - 1) + 1) && < 1. \\
-1 <& a_{n,k} && -\mathfrak{d}_{a_{n,k-1},1} && < 1.
\end{aligned}$$

The two positive integer quantities $a_{n,k}$ and $\mathfrak{d}_{a_{n,k-1},1}$ have a difference of less than one thus proving the formula. □

Proof of Proposition 3.8 for ($\mathfrak{d}$)*:* Substitute $m = \mathfrak{d}_{n,k-1} = F_{2k-3}\kappa(n) + F_{2k-4}n + F_{2k-2}$ into (37) to obtain

$$0 \leq \phi(F_{2k-3}\kappa(n) + F_{2k-4}n + F_{2k-2}) - \kappa(F_{2k-3}\kappa(n) + F_{2k-4}n + F_{2k-2}) < 1.$$

Next, using (32), write this as

$$\begin{aligned}
0 \leq (F_{2k-2} - (-\tfrac{1}{\phi})^{2k-3})\kappa(n) + (F_{2k-3} - (-\tfrac{1}{\phi})^{2k-4})n + (F_{2k-1} - (-\tfrac{1}{\phi})^{2k-2}) \\
- \kappa(F_{2k-3}\kappa(n) + F_{2k-4}n + F_{2k-2}) < 1,
\end{aligned}$$

or

$$\begin{aligned}
&(-\tfrac{1}{\phi})^{2k-3}\kappa(n) + (-\tfrac{1}{\phi})^{2k-4}n + (-\tfrac{1}{\phi})^{2k-2} \\
&\quad \leq F_{2k-2}\kappa(n) + F_{2k-3}n + F_{2k-1} - \kappa(F_{2k-3}\kappa(n) + F_{2k-4}n + F_{2k-2}) \\
&\qquad < 1 + (-\tfrac{1}{\phi})^{2k-3}\kappa(n) + (-\tfrac{1}{\phi})^{2k-4}n + (-\tfrac{1}{\phi})^{2k-2}.
\end{aligned}$$

Simplify the lower bound $(-\frac{1}{\phi})^{2k-3}\kappa(n)+(-\frac{1}{\phi})^{2k-4}n+(-\frac{1}{\phi})^{2k-2} = -(\frac{1}{\phi})^{2k-3}\kappa(n)+(\frac{1}{\phi})^{2k-4}n+(\frac{1}{\phi})^{2k-2} = (\frac{1}{\phi})^{2k-4}(-(\frac{1}{\phi})\kappa(n)+n+(\frac{1}{\phi})^2)$, considering that the minimum of $n - \frac{1}{\phi}\kappa(n)$ occurs at $n = 0$, so that

$$\begin{aligned}
&\min_{n,k}(-\tfrac{1}{\phi})^{2k-3}\kappa(n) + (-\tfrac{1}{\phi})^{2k-4}n + (-\tfrac{1}{\phi})^{2k-2} \\
=&\lim_{k\to\infty}(\tfrac{1}{\phi})^{2k-2} \\
=&0.
\end{aligned}$$

Simplify the bound $1+(-\frac{1}{\phi})^{2k-3}\kappa(n)+(-\frac{1}{\phi})^{2k-4}n+(-\frac{1}{\phi})^{2k-2} = 1-(\frac{1}{\phi})^{2k-3}\kappa(n)+(\frac{1}{\phi})^{2k-4}n + (\frac{1}{\phi})^{2k-2} = 1 + (\frac{1}{\phi})^{2k-4}(-(\frac{1}{\phi})\kappa(n) + n + (\frac{1}{\phi})^2)$, considering that the maxima of $n - (\frac{1}{\phi})\kappa(n)$ occur at $n = F_{2m}$, so that

$$\begin{aligned}
&\max_{n,k} 1 + (-\tfrac{1}{\phi})^{2k-3}\kappa(n) + (-\tfrac{1}{\phi})^{2k-4}n + (-\tfrac{1}{\phi})^{2k-2}\\
=&\max_{m,k} 1 + (\tfrac{1}{\phi})^{2k-4}(-(\tfrac{1}{\phi})\kappa(F_{2m}) + F_{2m} + (\tfrac{1}{\phi})^2)\\
=&\max_{m,k} 1 + (\tfrac{1}{\phi})^{2k-4}(-(\tfrac{1}{\phi})(F_{2m+1} - 1) + F_{2m} + (\tfrac{1}{\phi})^2)\\
=&\max_{m,k} 1 + (\tfrac{1}{\phi})^{2k-4}(1 - (\tfrac{1}{\phi})^{2m+1})\\
=&\max_{k\geq 2} \lim_{m\to\infty} 1 + (\tfrac{1}{\phi})^{2k-4}(1 - (\tfrac{1}{\phi})^{2m+1})\\
=&\max_{k\geq 2} 1 + (\tfrac{1}{\phi})^{2k-4}\\
=&2,
\end{aligned}$$

where the one of the steps uses (32). Continue to manipulate the formula until the desired difference is bounded:

$$\begin{aligned}
0 <&F_{2k-2}\kappa(n) + F_{2k-3}n + F_{2k-1} - \kappa(F_{2k-3}\kappa(n) + F_{2k-4}n + F_{2k-2}) &&< 2,\\
0 <&F_{2k-1}\kappa(n) + F_{2k-2}n + F_{2k} \quad -[F_{2k-3}\kappa(n) + F_{2k-4}n + F_{2k-2}\\
&\qquad + \kappa(F_{2k-3}\kappa(n) + F_{2k-4}n + F_{2k-2})] &&< 2,\\
0 <&\mathfrak{d}_{n,k} \quad -(a_{\mathfrak{d}_{n,k-1},1} - 1) &&< 2,\\
-1 <&\mathfrak{d}_{n,k} \quad -a_{\mathfrak{d}_{n,k-1},1} &&< 1.
\end{aligned}$$

The two positive integer quantities $\mathfrak{d}_{n,k}$ and $a_{\mathfrak{d}_{n,k-1},1}$ have a difference of less than one thus proving the formula. □

Proof of Proposition 3.9 for (F)*:* For $k \geq 3$ take the claim through successive simplifications,

$$\begin{aligned}
\mathsf{F}_{n,k} &= \llcorner_{\mathsf{F}_{n,k-2},1}\\
n + F_{F^{-1}(n)+k+1} &= n + F_{F^{-1}(n)+k-1} + F_{F^{-1}(n+F_{F^{-1}(n)+k-1})+3}\\
&\quad - F_{F^{-1}(n+F_{F^{-1}(n)+k-1})+2}\\
F_{F^{-1}(n)+k} &= F_{F^{-1}(n+F_{F^{-1}(n)+k-1})+1}\\
F^{-1}(n) + k &= F^{-1}(n + F_{F^{-1}(n)+k-1}) + 1\\
F^{-1}(n) + (k-2) + 1 &= F^{-1}(n + F_{F^{-1}(n)+(k-2)+1})
\end{aligned}$$

where (41) showed the latter for $k - 2 \geq 1$. □

Proof of Proposition 3.9 for (L)*:* For $k \geq 3$ take the claim through successive simplifications,

$$\begin{aligned}
\mathsf{L}_{n,k} &= \mathsf{F}_{\mathsf{L}_{n,k-2},1} \\
n + F_{F^{-1}(n)+k+2} - F_{F^{-1}(n)+2} &= n + F_{F^{-1}(n)+k} - F_{F^{-1}(n)+2} \\
&\quad + F_{F^{-1}(n+F_{F^{-1}(n)+k}-F_{F^{-1}(n)+2})+2} \\
F_{F^{-1}(n)+k+1} &= F_{F^{-1}(n+F_{F^{-1}(n)+k}-F_{F^{-1}(n)+2})+2} \\
F^{-1}(n) + k + 1 &= F^{-1}(n + F_{F^{-1}(n)+k} - F_{F^{-1}(n)+2}) + 2 \\
F^{-1}(n) + (k-2) + 1 &= F^{-1}(n + F_{F^{-1}(n)+k} - F_{F^{-1}(n)+2}) \\
F^{-1}(n) + (k-2) + 1 &= F^{-1}(\mathsf{L}_{n,k-2})
\end{aligned}$$

where (41) showed the latter for $k - 2 \geq 1$. $\square$

Proof of Proposition 3.9 for (w)*:* Beginning with (37), take $m = n + 1$ and rearrange to obtain

$$0 < 1 - \tfrac{1}{\phi} < \tfrac{1}{\phi}\kappa(n+1) - n \leq 1,$$

and in particular for $k \geq 3$,

$$0 < (\tfrac{1}{\phi})^{k-2} - (\tfrac{1}{\phi})^{k-1} < (\tfrac{1}{\phi})^{k-1}\kappa(n+1) - (\tfrac{1}{\phi})^{k-2}n \leq (\tfrac{1}{\phi})^{k-2} < 1,$$

and further, for the case $k \geq 3$ odd,

$$0 < (-\tfrac{1}{\phi})^{k-1}\kappa(n+1) + (-\tfrac{1}{\phi})^{k-2}n < 1. \tag{43}$$

Return to (37), with $m = w_{n,k-2} = F_{k-1}\kappa(n+1) + F_{k-2}n$:

$$0 < \phi F_{k-1}\kappa(n+1) + \phi F_{k-2}n - \kappa(F_{k-1}\kappa(n+1) + F_{k-2}n) \leq 1,$$

or, using (32),

$$0 < (F_k - (-\tfrac{1}{\phi})^{k-1})\kappa(n+1) + (F_{k-1} - (-\tfrac{1}{\phi})^{k-2})n - \kappa(F_{k-1}\kappa(n+1) + F_{k-2}n) \leq 1,$$

or

$$\begin{aligned}
&(-\tfrac{1}{\phi})^{k-1}\kappa(n+1) + (-\tfrac{1}{\phi})^{k-2}n \\
&\qquad < F_k\kappa(n+1) + F_{k-1}n - \kappa(F_{k-1}\kappa(n+1) + F_{k-2}n) \\
&\qquad\qquad \leq 1 + (-\tfrac{1}{\phi})^{k-1}\kappa(n+1) + (-\tfrac{1}{\phi})^{k-2}n.
\end{aligned}$$

Supposing that $k \geq 3$ is odd and using (43), this gives

$$\begin{aligned}
0 &< F_k\kappa(n+1) + F_{k-1}n - \kappa(F_{k-1}\kappa(n+1) + F_{k-2}n) < 2, \\
-1 &< F_{k+1}\kappa(n+1) + F_k n - (\kappa(F_{k-1}\kappa(n+1) + F_{k-2}n) \\
&\qquad + F_{k-1}\kappa(n+1) + F_{k-1}n + 1) < 1, \\
-1 &< w_{n,k} - a_{w_{n,k-2},1} < 1.
\end{aligned}$$

Thus positive integer quantities $w_{n,k}$ and $a_{w_{n,k-2},1}$ must be equal for $k \geq 3$ odd.

Analogously to (43), for the case $k \geq 4$ even, write

$$0 < (\tfrac{1}{\phi})^{k-3} - (\tfrac{1}{\phi})^{k-2} < (-\tfrac{1}{\phi})^{k-2}\kappa(n+1) + (-\tfrac{1}{\phi})^{k-3}n \leq (\tfrac{1}{\phi})^{k-3} < 1. \tag{44}$$

Return to (37), with $m = w_{n,k-3} = F_{k-2}\kappa(n+1) + F_{k-3}n$:

$$0 < \phi F_{k-2}\kappa(n+1) + \phi F_{k-3}n - \kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) \leq 1,$$

or, using (32),

$$0 < (F_{k-1} - (-\tfrac{1}{\phi})^{k-2})\kappa(n+1) + (F_{k-2} - (-\tfrac{1}{\phi})^{k-3})n - \kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) \leq 1,$$

or

$$\begin{aligned}(-\tfrac{1}{\phi})^{k-2}\kappa(n+1) + (-\tfrac{1}{\phi})^{k-3}n \\ < F_{k-1}\kappa(n+1) + F_{k-2}n - \kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) \\ \leq 1 + (-\tfrac{1}{\phi})^{k-2}\kappa(n+1) + (-\tfrac{1}{\phi})^{k-3}n.\end{aligned}$$

Supposing that $k \geq 4$ is even and using (44), this gives

$$\begin{aligned} 0 <& F_{k-1}\kappa(n+1) + F_{k-2}n - \kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) &< 2, \\ -1 <& F_{k-1}\kappa(n+1) + F_{k-2}n - (\kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) + 1) &< 1. \end{aligned}$$

Thus positive integer quantities $w_{n,k-2}$ and $\kappa(w_{n,k-3}) + 1$ must be equal for $k \geq 4$:

$$\begin{aligned} F_{k-1}\kappa(n+1) + F_{k-2}n =& \ \kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) + 1, \\ 2F_{k-1}\kappa(n+1) + 2F_{k-2}n =& 2\kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) + 2, \\ (F_{k+1} - F_{k-2})\kappa(n+1) + (F_k - F_{k-3})n =& 2\kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) + 2, \\ F_{k+1}\kappa(n+1) + F_k n =& 2\kappa(F_{k-2}\kappa(n+1) + F_{k-3}n) \\ & + F_{k-2}\kappa(n+1) + F_{k-3}n + 2, \\ w_{n,k} =& \ a_{w_{n,k-3},2}. \end{aligned}$$

□

Proof of Proposition 3.9 for (*a*)*:* Beginning with (37), take $m = n$ and rearrange to obtain

$$0 \leq (\tfrac{1}{\phi})^{k-2}n - (\tfrac{1}{\phi})^{k-1}\kappa(n) < (\tfrac{1}{\phi})^{k-2},$$

and in particular for $k \geq 3$

$$-(\tfrac{1}{\phi})^{k-2} < (\tfrac{1}{\phi})^{k-1}\kappa(n) - (\tfrac{1}{\phi})^{k-2}n \leq 0,$$

and further, for the case $k \geq 3$ odd,

$$-(\tfrac{1}{\phi})^{k-2} < (-\tfrac{1}{\phi})^{k-1}\kappa(n) + (-\tfrac{1}{\phi})^{k-2}n \leq 0,$$

or,

$$\begin{aligned} -1 < -(\tfrac{1}{\phi})^{k-2} - (\tfrac{1}{\phi})^k \\ < (-\tfrac{1}{\phi})^{k-1}\kappa(n) + (-\tfrac{1}{\phi})^{k-2}n + (-\tfrac{1}{\phi})^k \\ \leq -(\tfrac{1}{\phi})^k < 1, \text{ for } k \geq 3 \text{ odd.} \end{aligned} \tag{45}$$

Return to (37), with $m = a_{n,k-2} + 1 = F_{k-1}\kappa(n) + F_{k-2}n + F_k$:

$$0 < \phi F_{k-1}\kappa(n) + \phi F_{k-2}n + F_k - \kappa(F_{k-1}\kappa(n) + F_{k-2}n + F_k) \leq 1,$$

or, using (32),

$$\begin{aligned} 0 < (F_k - (-\tfrac{1}{\phi})^{k-1})\kappa(n) + (F_{k-1} - (-\tfrac{1}{\phi})^{k-2})n + (F_{k+1} - (-\tfrac{1}{\phi})^k)F_k \\ - \kappa(F_{k-1}\kappa(n) + F_{k-2}n + F_k) \leq 1, \end{aligned}$$

or

$$
\begin{aligned}
&(-\tfrac{1}{\phi})^{k-1}\kappa(n) + (-\tfrac{1}{\phi})^{k-2}n + (-\tfrac{1}{\phi})^{k} \\
&\qquad < F_k\kappa(n) + F_{k-1}n + F_{k+1} - \kappa(F_{k-1}\kappa(n) + F_{k-2}n + F_k) \\
&\qquad\qquad \le 1 + (-\tfrac{1}{\phi})^{k-1}\kappa(n) + (-\tfrac{1}{\phi})^{k-2}n + (-\tfrac{1}{\phi})^{k}.
\end{aligned}
$$

Supposing that $k \ge 3$ is odd and using (45), this gives

$$
\begin{aligned}
-1 <{}& F_k \quad \kappa(n) + F_{k-1}n + F_{k+1} \qquad -\kappa(F_{k-1}\kappa(n) + F_{k-2}n + F_k) \qquad < 1, \\
-1 <{}& F_{k+1}\kappa(n) + F_k \quad n + F_{k+2} - 1 - (\kappa(F_{k-1}\kappa(n) + F_{k-2}n + F_k) \\
&\qquad + \quad F_{k-1}\kappa(n) + F_{k-2}n + F_k \ -1) < 1, \\
&-1 < a_{n,k} - w_{a_{n,k-2},1} < 1.
\end{aligned}
$$

Thus positive integer quantities $a_{n,k}$ and $w_{a_{n,k-2},1}$ must be equal for $k \ge 3$ odd.
For the case $k \ge 4$ even, similarly to (45), write

$$
\begin{aligned}
-1 &< -(\tfrac{1}{\phi})^{k-3} - (\tfrac{1}{\phi})^{k-1} \\
&< (-\tfrac{1}{\phi})^{k-2}\kappa(n) + (-\tfrac{1}{\phi})^{k-3}n + (-\tfrac{1}{\phi})^{k-1} \\
&\le -(\tfrac{1}{\phi})^{k-1} < 1.
\end{aligned}
\tag{46}
$$

Return to (37), with $m = a_{n,k-3} + 1 = F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}$:

$$
0 < \phi F_{k-2}\kappa(n) + \phi F_{k-3}n + F_{k-1} - \kappa(F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}) \le 1,
$$

or, using (32),

$$
\begin{aligned}
0 < (F_{k-1} - (-\tfrac{1}{\phi})^{k-2})\kappa(n) + (F_{k-2} - (-\tfrac{1}{\phi})^{k-3})n + F_k - (-\tfrac{1}{\phi})^{k-1} \\
- \kappa(F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}) \le 1,
\end{aligned}
$$

or

$$
\begin{aligned}
&(-\tfrac{1}{\phi})^{k-2}\kappa(n) + (-\tfrac{1}{\phi})^{k-3}n + (-\tfrac{1}{\phi})^{k-1} \\
&\qquad < F_{k-1}\kappa(n) + F_{k-2}n + F_k - \kappa(F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}) \\
&\qquad\qquad \le 1 + (-\tfrac{1}{\phi})^{k-2}\kappa(n) + (-\tfrac{1}{\phi})^{k-3}n + (-\tfrac{1}{\phi})^{k-1}.
\end{aligned}
$$

Supposing that $k \ge 4$ is even and using (46), this gives

$$
-1 < F_{k-1}\kappa(n) + F_{k-2}n + F_k - \kappa(F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}) < 1.
$$

Thus positive integer quantities $a_{n,k-2} + 1$ and $\kappa(a_{n,k-3} + 1)$ must be equal for $k \ge 4$ even:

$$
\begin{aligned}
F_{k-1}\kappa(n) + F_{k-2}n + F_k &= \ \kappa(F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}), \\
2F_{k-1}\kappa(n) + 2F_{k-2}n + 2F_k &= 2\kappa(F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}), \\
(F_{k+1} - F_{k-2})\kappa(n) + (F_k - F_{k-3})n & \\
+(F_{k+2} - F_{k-1}) - 1 &= 2\kappa(F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}) - 1, \\
F_{k+1}\kappa(n) + F_k n + F_{k+2} - 1 &= 2\kappa(F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1}) \\
&\quad + F_{k-2}\kappa(n) + F_{k-3}n + F_{k-1} - 1, \\
a_{n,k} &= w_{a_{n,k-3},2} \qquad .
\end{aligned}
$$

□

Proposition 4.6 (Interspersion Property (I4) of $\boldsymbol{a}$)**.** *Start with element $a_{n,k}$, for some $n = 0, 1, 2, \ldots$, $k = 1, 2, 3 \ldots$. Without loss of generality, let interspersions of rows indexed n and N, with $N > n$, take the form*

$$\begin{aligned} & a_{n,k} < a_{N,1} < a_{n,k+1} < a_{N,2} < a_{n,k+2} < \cdots \\ & \qquad \cdots < a_{n,k+h-1} < a_{N,h} < a_{n,k+h} < \cdots , \end{aligned} \tag{47}$$

where column offset $h = 1, 2, 3 \ldots$. The following statements then hold:

(i) Least N for which rows n and N intersperse starting with $a_{n,k}$: Suppose $N = a_{n,k-2} + 1 \leq a_{n,k-1}$. Then (47) holds with

$$a_{N,h} - a_{n,k+h-1} = \begin{cases} F_h, & \text{if } k \geq 2 \text{ even;} \\ F_{h+2}, & \text{if } k \geq 3 \text{ odd;} \\ F_{h+2}, & \text{if } k = 1 \text{ and } n \in \{0\} \cup \Lambda; \\ F_{h+3}, & \text{if } k = 1 \text{ and } n \in K. \end{cases} \tag{48}$$

(ii) Greatest N for which rows n, N intersperse starting with $a_{n,k}$: Suppose $N = a_{n,k-1} \geq a_{n,k-2} + 1$. Then (47) holds with

$$a_{n,k+h} - a_{N,h} = \begin{cases} F_{h+1}, & \text{if } k \geq 2; \\ 2F_{h+1}, & \text{if } k = 1. \end{cases} \tag{49}$$

(iii) Range of N for which rows n and N intersperse starting with $a_{n,k}$ — Necessary Condition: If $a_{n,k-1} \geq a_{n,k-2} + 1$, then (47) holds for any of the $a_{n,k-3} + 1$ rows $N \in \{a_{n,k-2} + 1, a_{n,k-2} + 2, \ldots, a_{n,k-1} - 1, a_{n,k-1}\}$.

(iv) Range of N for which rows n and N intersperse starting with $a_{n,k}$ — Sufficient Condition: If (47) holds for some N, then $a_{n,k-1} \geq a_{n,k-2} + 1$ and $N \in \{a_{n,k-2} + 1, a_{n,k-2} + 2, \ldots, a_{n,k-1} - 1, a_{n,k-1}\}$.

Proof. Proceding one statement at a time.

(i) First statement, $N = a_{n,k-2} + 1$: Recall from Corollary 3.3 that $a_{n,k} = a_{n,k-1} + a_{n,k-2} + 1$, so the sequence of differences $(a_{N,1} - a_{n,k}, a_{N,2} - a_{n,k+1}, \ldots, a_{N,h} - a_{n,k+h-1}, \ldots)$ satisfies the Fibonacci recurrence. Hence, it is sufficient to quantify the first two differences $a_{N,1} - a_{n,k}$ and $a_{N,2} - a_{n,k+1}$. These quantities will be shown to be, respectively,

$$a_{N,1} - a_{n,k} = \begin{cases} 1, & \text{if } k \geq 2 \text{ even;} \\ 2, & \text{if } k \geq 3 \text{ odd;} \\ 2, & \text{if } k = 1 \text{ and } n \in \{0\} \cup \Lambda; \\ 3, & \text{if } k = 1 \text{ and } n \in K; \text{ and} \end{cases} \tag{50}$$

$$a_{N,2} - a_{n,k+1} = \begin{cases} 1, & \text{if } k \geq 2 \text{ even;} \\ 3, & \text{if } k \geq 3 \text{ odd;} \\ 3, & \text{if } k = 1 \text{ and } n \in \{0\} \cup \Lambda; \\ 5, & \text{if } k = 1 \text{ and } n \in K. \end{cases} \tag{51}$$

First consider $a_{N,1} - a_{n,k}$. From the cohort-based formula (1), observe that, in particular, $a_{N,1} = \lfloor N\phi \rfloor + N + 1$. Hence, $a_{N,1} - a_{n,k} = (\lfloor N\phi \rfloor + N + 1) - a_{n,k}$. Selectively substituting $a_{n,k-2} + 1$ for N, rearrange this difference into $\phi + 2 - (N\phi - \lfloor N\phi \rfloor) - (a_{n,k} - a_{n,k-2}\phi^2)$ and examine the bounds on the terms. The quantity $N\phi - \lfloor N\phi \rfloor$, an instance of (37), must lie in the interval $[0, 1)$. By Proposition 4.2,

$$a_{n,k} - a_{n,k-2}\phi^2 \in \begin{cases} [\phi^2 + 1/\phi^k - 1, \phi^2 + 1/\phi^{k-2} - 1), & k \text{ even;} \\ (\phi^2 - 1/\phi^{k-2} - 1, \phi^2 - 1/\phi^k - 1], & k \text{ odd.} \end{cases}$$

So we find that for $k \geq 2$ even, $a_{N,1} - a_{n,k} \in (1 - 1/\phi^{k-2}, 2 - 1/\phi^k) \subset (0, 2)$, which, being an integer, must therefore equal 1. For $k \geq 1$ odd, we find that $a_{N,1} - a_{n,k} \in (1 + 1/\phi^k, 2 + 1/\phi^{k-2})$, which for $k \geq 3$ is a subset of $(1, 3)$ and must therefore equal 2. For $k = 1$, however, the bounds $(1 + 1/\phi, 2 + \phi) \subset (1, 4)$ allow for a difference of either 2 or 3. Thus, in this case, the bounds are insufficiently tight, requiring a different approach.

For $k = 1$, consider $a_{N,1} - a_{n,k} = a_{a_{n,k-2}+1,1} - a_{n,k}$ directly, and use (1) to obtain $a_{a_{n,1-2}+1,1} - a_{n,1} = a_{a_{n,-1}+1,1} - a_{n,1} = \lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor + 1$. The latter expression is a Fibonacci word that takes the value 3 on the Lower Wythoff numbers K, and 2 at 0 and on the Upper Wythoff numbers Λ, which completes the proof of (50). Analogous steps prove (51), from which (48) obtains by the Fibonacci recurrence.

To complete the proof of (47) for $N = a_{n,k-2} + 1$ requires the remaining inequalities, namely $a_{n,k+1} - a_{N,1} \geq 1, a_{n,k+2} - a_{N,2} \geq 1, \ldots, a_{n,k+h} - a_{N,h} \geq 1, \ldots$, to hold. However, these will be shown next for the larger row index $N = a_{n,k-1}$ of the second statement, and since the columns of $(a_{N,k})$ strictly increase with N, it must be the case that $a_{n,k+h} - a_{a_{n,k-2}+1,h} > a_{n,k+h} - a_{a_{n,k-1},h} \geq 1$, so long as the assumption $a_{n,k-2} + 1 \leq a_{n,k-1}$ is met.

(ii) Second statement, $N = a_{n,k-1}$: To prove (47) for $N = a_{n,k-1}$, first observe that all inequalities $a_{N,1} - a_{n,k} \geq 1, a_{N,2} - a_{n,k+1} \geq 1, \ldots, a_{N,h} - a_{n,k+h-1} \geq 1, \ldots$ follow from (i), where they are proved for the smaller row index $N = a_{n,k-2} + 1$. Whereas the columns of $(a_{N,k})$ are increasing in N, it must be the case that $a_{a_{n,k-1},h} - a_{n,k+h-1} > a_{a_{n,k-2}+1,h} - a_{n,k+h-1} \geq 1$, provided the assumption $a_{n,k-1} \geq a_{n,k-2} + 1$ is met.

It remains to prove the inequalities $a_{n,k+1} - a_{N,1} \geq 1$, $a_{n,k+2} - a_{N,2} \geq 1, \ldots$, $a_{n,k+h} - a_{N,h} \geq 1, \ldots$. Similar to the proof of (i), begin by noting that the sequence of differences $(a_{n,k+1} - a_{N,1}, a_{n,k+2} - a_{N,2}, \ldots, a_{n,k+h} - a_{N,h}, \ldots)$ satisfies the Fibonacci recurrence, thus making it sufficient to quantify the first two differences $a_{n,k+1} - a_{N,1}$ and $a_{n,k+2} - a_{N,2}$. These quantities will be shown to be, respectively,

$$a_{n,k+1} - a_{N,1} = \begin{cases} 1, & \text{if } n = 0 \text{ or } k \geq 2; \\ 2, & \text{if } n \geq 1 \text{ and } k = 1; \text{ and} \end{cases} \tag{52}$$

$$a_{n,k+2} - a_{N,2} = \begin{cases} 2, & \text{if } n = 0 \text{ or } k \geq 2; \\ 4, & \text{if } n \geq 1 \text{ and } k = 1. \end{cases} \tag{53}$$

Begin with $a_{n,k+1} - a_{N,1}$. From (1), $a_{N,1} = \lfloor N\phi \rfloor + N + 1$, hence, $a_{n,k+1} - a_{N,1} = a_{n,k+1} - (\lfloor N\phi \rfloor + N + 1)$. Selectively substituting $a_{n,k-1}$ for N, we rearrange this difference into $(N\phi - \lfloor N\phi \rfloor) + (a_{n,k+1} - a_{n,k-1}\phi^2) - 1$ and examine the bounds on the terms. The quantity $N\phi - \lfloor N\phi \rfloor$, an instance of (37), must lie in the interval $[0, 1)$. By Proposition 4.2,

$$a_{n,k+1} - a_{n,k-1}\phi^2 \in \begin{cases} [\phi^2 + 1/\phi^{k+1} - 1, \phi^2 + 1/\phi^{k-1} - 1), & k \text{ odd;} \\ (\phi^2 - 1/\phi^{k-1} - 1, \phi^2 - 1/\phi^{k+1} - 1], & k \text{ even.} \end{cases}$$

So we find that for $k \geq 2$ even, $a_{n,k+1} - a_{N,1} \in (\phi^2 - 1/\phi^{k-1} - 2, \phi^2 - 1/\phi^{k+1} - 1) \subset (0,2)$, which, being an integer, must therefore equal 1. For k odd, we find that $a_{N,1} - a_{n,k} \in [\phi^2 + 1/\phi^{k+1} - 2, \phi^2 + 1/\phi^{k-1} - 1)$, which for $k \geq 3$ is a subset of $(0,2)$ and must therefore equal 1. For $k = 1$, however, the bounds $[\phi^2 + 1/\phi^2 - 2, \phi^2) \subset [1,3)$ allow for a difference of either 1 or 2. Thus, in this case the bounds are insufficiently tight, requiring a different approach.

To improve the lower bound for $k = 1$, apply the cohort-based formula (1), to obtain $a_{n,k+1} - a_{n,k-1}\phi^2 = a_{n,2} - a_{n,0}\phi^2 = n - \frac{\lfloor n\phi \rfloor}{\phi} + 2$, which by (38) implies that $2 \leq a_{n,2} - a_{n,0}\phi^2 < \phi^2$, with the first inequality strict for $n > 0$. Consequently, for $n > 0$, we have $a_{n,2} - a_{n,0}\phi^2 \in (2,3)$ thus implying $a_{n,2} - a_{a_{n,0},1} = 2$. For $n = 0$ and $k = 1$, direct evaluation gives $a_{n,k+1} - a_{a_{n,k-1},1} = a_{0,2} - a_{a_{0,0},1} = 1$, so proving the last case of (52). Analogous steps prove (53), and from these two, the Fibonacci recurrence gives

$$a_{n,k+1} - a_{N,1} = \begin{cases} F_{h+1}, & \text{if } n = 0 \text{ or } k \geq 2; \\ 2F_{h+1}, & \text{if } n \geq 1 \text{ and } k = 1. \end{cases}$$

Note that the conditions of (49) were simplified using the assumption $a_{n,k-1} - (a_{n,k-2} + 1) = a_{n,k-3} \geq 0$. Observe that the only nonnegative n and positive k to falsify the assumption are $(n,k) \in \{(0,1),(1,1)\}$. Put differently, the first two rows are the only rows that do not intersperse entirely with another row, because the gaps between the first and second elements, $a_{0,2} - a_{0,1}$, respectively, $a_{1,2} - a_{1,1}$ are too small to allow this interspersion. See Remark 3.4 for a visualization of this statement.

(iii) Third statement, sufficiency: Sufficiency has already been demonstrated in (i) by the fact that inequalities $a_{N,1} - a_{n,k} \geq 1, a_{N,2} - a_{n,k+1} \geq 1, \ldots, a_{N,h} - a_{n,k+h-1} \geq 1, \ldots$ hold for $N \geq a_{n,k-2}+1$ and in (ii) by the fact that inequalities $a_{n,k+1} - a_{N,1} \geq 1, a_{n,k+2} - a_{N,2} \geq 1, \ldots, a_{n,k+h} - a_{N,h} \geq 1, \ldots$ hold for $N \leq a_{n,k-1}$.

(iv) Fourth statement, necessity: For this it remains to show that inequalities $a_{n,k+1} - a_{N,1} \geq 1, a_{n,k+2} - a_{N,2} \geq 1, \ldots, a_{n,k+h} - a_{N,h} \geq 1, \ldots$ fail to hold for $N > a_{n,k-1}$ and that inequalities $a_{N,1} - a_{n,k} \geq 1, a_{N,2} - a_{n,k+1} \geq 1, \ldots, a_{N,h} - a_{n,k+h-1} \geq 1, \ldots$ fail to hold for $N < a_{n,k-2} + 1$. Whereas columns of $(a_{N,k})$ increase in N, it suffices to show these for $N = a_{n,k-1} + 1$, respectively, $N = a_{n,k-2}$.

Begin with $a_{n,k+h} - a_{N,h}$ for $N = a_{n,k-1}+1$. From (i), already proven, observe that (48) implies $a_{n,k+h-1} - a_{a_{n,k-2}+1,h} \leq -1$ for $k = 1,2,3,\ldots$. The substitution $j = k-1$, shows that $a_{n,j+h} - a_{a_{n,j-1}+1,h} \leq -1$ for $j = 0,1,2,\ldots$, in particular for $j = 1,2,3,\ldots$.

Similarly take $a_{N,h} - a_{n,k+h-1}$ for $N = a_{n,k-2}$. From (ii), already proven, observe that (49) implies $a_{a_{n,k-1},h} - a_{n,k+h} \leq -1$ for $k = 1,2,3,\ldots$. The substitution $j = k+1$, shows that $a_{a_{n,j-2},h} - a_{n,j+h-1} \leq -1$ for $j = 2,3,4,\ldots$. Thus it remains only to demonstrate that $a_{a_{n,j-2},h} - a_{n,j+h-1} \ngeq 1$ for $j = 1$, or $a_{a_{n,-1},h} - a_{n,h} \ngeq 1$, which follows from (1), since $a_{n,-1} = n$, thus $a_{a_{n,-1},h} - a_{n,h} = 0$.

□

Proposition 4.7 (Interspersion Property (I4) of $\boldsymbol{b}$)**.** *For $n = 0,1,2,\ldots$ and $k = 1,2,3\ldots$, consider element $b_{n,k}$. Without loss of generality, let interspersions start with element $b_{n,k}$, and consider both strict interspersion*

$$b_{n,k} < b_{N,1} < b_{n,k+1} < b_{N,2} < b_{n,k+2} < \cdots \\ \cdots < b_{n,k+h-1} < b_{N,h} < b_{n,k+h} < \cdots, \tag{54}$$

of pairs of rows indexed n and N, with $n < N$ and as well as coincidence

$$\begin{aligned} b_{n,k} = b_{N,1} < b_{n,k+1} = b_{N,2} < b_{n,k+2} \leq \cdots \\ \cdots < b_{n,k+h-1} = b_{N,h} < b_{n,k+h} \leq \cdots , \end{aligned} \tag{55}$$

of two rows $n \leq N$, where $h = 1, 2, 3 \ldots$ is a column offset.

(i) Least N for which rows n and N coincide starting with $b_{n,k}$: *Suppose $N = b_{n,k-2} + 1$, then (55) holds.*

(ii) Least N for which rows n and N intersperse starting with $b_{n,k}$: *Suppose $N = b_{n,k-2} + 2$. Then*

$$b_{N,h} - b_{n,k+h-1} = \begin{cases} F_{h+2}, & \textit{if } k \textit{ even and } n \in K; \\ F_{h+2}, & \textit{if } k \textit{ odd and } n \in \{0\} \cup \Lambda; \\ F_{h+3}, & \textit{if } k \textit{ odd and } n \in K; \\ F_{h+3}, & \textit{if } k \textit{ even and } n \in \{0\} \cup \Lambda; \end{cases}$$

Moreover, if $b_{n,k-2} + 2 \leq b_{n,k-1}$, then (54) holds.

(iii) Greatest N for which rows n, N intersperse starting with $b_{n,k}$: *Suppose $N = b_{n,k-1} \geq b_{n,k-2} + 2$. Then (54) holds with*

$$b_{n,k+h} - b_{N,h} = F_{h+3}. \tag{56}$$

(iv) Range of N for which rows n and N intersperse starting with $b_{n,k}$: — Necessary Condition: *If $b_{n,k-1} \geq b_{n,k-2} + 2$, then (54) holds for any of the $b_{n,k-3} + 1$ rows $N \in \{b_{n,k-2} + 2, b_{n,k-2} + 3, \ldots, b_{n,k-1} - 1, b_{n,k-1}\}$.*

(v) Range of N for which rows n and N intersperse starting with $b_{n,k}$: — Sufficient Condition: *If (54) holds for some N, then $b_{n,k-1} \geq b_{n,k-2} + 2$ and $N \in \{b_{n,k-2} + 2, b_{n,k-2} + 2, \ldots, b_{n,k-1} - 1, b_{n,k-1}\}$.*

Proof. Similar to that of Proposition 4.6. Note that for $N = b_{n,k-1}$,

$$b_{n,k+h} - b_{N,h} = \begin{cases} F_{h+2}, & \text{if } n = 0 \text{ and } k = 1; \\ F_{h+3}, & \text{otherwise;} \end{cases}$$

which simplifies using $b_{n,k-1} \geq b_{n,k-2} + 2$ to give (56). $\square$

Proposition 4.8 (Interspersion Property (I4) of $\boldsymbol{c}$)**.** *For $n = 0, 1, 2, \ldots$ and $k = 1, 2, 3 \ldots$, consider element $c_{n,k}$. Without loss of generality, let interspersions start with element $c_{n,k}$, and consider both strict interspersion*

$$\begin{aligned} c_{n,k} < c_{N,1} < c_{n,k+1} < c_{N,2} < c_{n,k+2} < \cdots \\ \cdots < c_{n,k+h-1} < c_{N,h} < c_{n,k+h} < \cdots , \end{aligned} \tag{57}$$

of pairs of rows indexed n and N, with $n < N$ and as well as coincidence

$$\begin{aligned} c_{n,k} = c_{N,1} < c_{n,k+1} = c_{N,2} < c_{n,k+2} \leq \cdots \\ \cdots < c_{n,k+h-1} = c_{N,h} < c_{n,k+h} \leq \cdots , \end{aligned} \tag{58}$$

of two rows $n \leq N$, where $h = 1, 2, 3 \ldots$ is a column offset.

(i) Least N for which rows n and N coincide starting with $c_{n,k}$: *Let $N = c_{n,k-3} + 1$. Then for $k = 1, 3, 5, \ldots$ odd, (58) holds.*

(ii) Least N for which rows n and N intersperse starting with $c_{n,k}$: Let $N = c_{n,k-3}+1$. Then for $k = 2, 4, 6, \ldots$ even, $c_{N,h} - c_{n,k+h-1} = F_{h+2}$. Moreover, if $c_{n,k-3} + 1 \leq c_{n,k-2}$, then (57) holds. Suppose $N = c_{n,k-3} + 2$. Then

$$c_{N,h} - c_{n,k+h-1} = \begin{cases} F_{h+3}, & \text{if } k = 1 \quad \text{and } n \in \{0\} \cup \Lambda; \\ L_{h+3}, & \text{if } k = 2 \quad \text{and } n \in \{0\} \cup \Lambda; \\ F_{h+4}, & \text{if } k = 1, 2 \text{ and } n \in K; \\ F_{h+4}, & \text{if } k \geq 3. \end{cases}$$

Moreover, if $c_{n,k-3} + 2 \leq c_{n,k-2}$, then (57) holds.

(iii) Greatest N for which rows n, N intersperse starting with $c_{n,k}$: Suppose $N = c_{n,k-2} \leq c_{n,k-2}$. Then (57) holds with

$$c_{n,k+h} - c_{N,h} = F_{h+3}. \tag{59}$$

(iv) Range of N for which rows n and N intersperse starting with $c_{n,k}$: — Necessary Condition: If $c_{n,k-2} \geq c_{n,k-3} + 2$, then (57) holds for any of the $c_{n,k-4}$ rows $N \in \{c_{n,k-3} + 2, c_{n,k-3} + 2, \ldots, c_{n,k-2} - 1, c_{n,k-2}\}$.

(v) Range of N for which rows n and N intersperse starting with $c_{n,k}$: — Sufficient Condition: If (57) holds for some N, then $c_{n,k-2} \geq c_{n,k-3} + 2$ and $N \in \{c_{n,k-3} + 2, c_{n,k-3} + 2, \ldots, c_{n,k-2} - 1, c_{n,k-2}\}$.

Proof. Similar to that of Proposition 4.6. Note that for $N = c_{n,k-2}$,

$$c_{n,k+h} - c_{N,h} = \begin{cases} F_{h+1}, & \text{if } n = 0 \text{ and } k = 1; \\ F_{h+3}, & \text{otherwise}; \end{cases}$$

which simplifies using $c_{n,k-3} + 2 \leq c_{n,k-2}$ to give (59). □

Proposition 4.9 (Interspersion Property (I4) of $\boldsymbol{d}$). *Start with element $d_{n,k}$, for some $n = 0, 1, 2, \ldots$, $k = 1, 2, 3 \ldots$. Without loss of generality,* strict *interspersions of rows indexed n and N, with $N > n$, take the form*

$$\begin{aligned} d_{n,k} < d_{N,1} < d_{n,k+1} < d_{N,2} < d_{n,k+2} < \cdots \\ \cdots < d_{n,k+h-1} < d_{N,h} < d_{n,k+h} < \cdots, \end{aligned} \tag{60}$$

where column offset $h = 1, 2, 3 \ldots$. The following statements then hold:

(i) Least N for which rows n and N intersperse starting with $d_{n,k}$: Suppose $N = d_{n,k-3} + 2 \leq d_{n,k-2} + 1$. Then (60) holds and, in particular,

$$d_{N,h} - d_{n,k+h-1} = \begin{cases} F_{h+1}, & \text{if } k \geq 2 \text{ even}; \\ F_{h+3}, & \text{if } k \geq 3 \text{ odd}; \\ F_{h+3}, & \text{if } k = 1 \text{ and } n \in \{0\} \cup \Lambda; \\ F_{h+4}, & \text{if } k = 1 \text{ and } n \in K; \end{cases}$$

(ii) Greatest N for which rows n, N intersperse starting with $d_{n,k}$: Suppose $N = d_{n,k-2} + 1 \geq d_{n,k-3} + 2$. Then (60) holds and

$$d_{n,k+h} - d_{N,h} = \begin{cases} F_{h+2}, & \text{if } k \geq 2; \\ 2F_{h+2}, & \text{if } k = 1; \end{cases} \tag{61}$$

(iii) Range of N for which rows n and N intersperse starting with $d_{n,k}$: — Necessary Condition: If $d_{n,k-2} + 1 \geq d_{n,k-3} + 2$, then (60) holds for any of the $d_{n,k-4} + 2$ rows $N \in \{d_{n,k-3} + 2, d_{n,k-3} + 3, \ldots, d_{n,k-2} - 1, d_{n,k-2} + 1\}$.

(iv) Range of N for which rows n and N intersperse starting with $d_{n,k}$: — Sufficient Condition: If (60) holds for some N, then $d_{n,k-2}+1 \geq d_{n,k-3}+2$ and $N \in \{d_{n,k-3}+2, d_{n,k-3}+3, \ldots, d_{n,k-2}-1, d_{n,k-2}+1\}$.

Proof. Similar to that of Proposition 4.6. We note that for $N = d_{n,k-2}+1$,

$$d_{n,k+h} - d_{N,h} = \begin{cases} F_{h+2}, & \text{if } n = 0 \text{ or } k \geq 2; \\ 2F_{h+2}, & \text{if } n \geq 1 \text{ and } k = 1. \end{cases}$$

Simplify it using $d_{n,k-2}+1 \geq d_{n,k-3}+2$ to give (61). □

Proposition 4.10 (Interspersion Property (I4) of $\boldsymbol{w}$)**.** *Start with element $w_{n,k}$, for some $n = 0, 1, 2, \ldots$, $k = 1, 2, 3 \ldots$. Without loss of generality, let interspersions of rows indexed n and N, with $N > n$, take the form*

$$\begin{aligned} w_{n,k} < w_{N,1} < w_{n,k+1} < w_{N,2} < w_{n,k+2} < \cdots \\ \cdots < w_{n,k+h-1} < w_{N,h} < w_{n,k+h} < \cdots, \end{aligned} \tag{62}$$

where column offset $h = 1, 2, 3 \ldots$. The following statements then hold:

(i) Least N for which rows n and N intersperse starting with $w_{n,k}$: For the first block ($k = 1$), suppose $N = w_{n,k-2}+1 \leq w_{n,k-1}-1$. Then (62) holds with

$$w_{N,h} - w_{n,k+h-1} = \begin{cases} L_{h+2} + F_{h+1}, & \textit{if } k = 2 \textit{ and } n \in \Lambda - 1; \\ L_{h+2}, & \textit{if } k \geq 4, \textit{ or } k = 2 \textit{ and } n \in K - 1; \\ F_{h+3}, & \textit{if } k = 3, \textit{ or } k = 1 \textit{ and } n \in K - 1; \\ F_{h+2}, & \textit{if } k = 1 \textit{ and } n \in \Lambda - 1. \end{cases}$$

For subsequent blocks ($k \geq 2$), suppose $N = w_{n,k-2} \leq w_{n,k-1}-1$. Then (62) holds with

$$w_{N,h} - w_{n,k+h-1} = \begin{cases} 2F_{h+1}, & \textit{if } k = 2; \\ F_{h+1}, & \textit{if } k \geq 3; \end{cases}$$

(ii) Greatest N for which rows n, N intersperse starting with $w_{n,k}$: Suppose $N = w_{n,k-1}-1 \geq w_{n,k-2}$ and $k \geq 2$ or $n \geq 1$. Then (62) holds with

$$w_{n,k+h} - w_{N,h} = \begin{cases} F_{h+2}, & \textit{if } k \textit{ even}; \\ F_h, & \textit{if } k \textit{ odd}. \end{cases}$$

(iii) Range of N for which rows n and N intersperse starting with $w_{n,k}$ — Necessary Condition: If $w_{n,k-1}-1 \geq w_{n,k-2}+1$ and $k \geq 1$, then (62) holds for any of the $w_{n,k-2}-1$ rows $N \in \{w_{n,k-2}+1, w_{n,k-2}+2, \ldots, w_{n,k-1}-2, w_{n,k-1}-1\}$, and further, if $k \geq 2$, then (62) also holds for the row $N = w_{n,k-2}$.

(iv) Range of N for which rows n and N intersperse starting with $w_{n,k}$ — Sufficient Condition: If (62) holds for some N, then either $k \geq 2$ and $w_{n,k-1}-1 \geq w_{n,k-2}$ and $N \in \{w_{n,k-2}, w_{n,k-2}+1, \ldots, w_{n,k-1}-2, w_{n,k-1}-1\}$, or $k = 1$ and $w_{n,0} - 1 \geq w_{n,-1}+1$ and $N \in \{w_{n,-1}+1, \ldots, w_{n,0}-1\}$

Proof. Similar to that of Proposition 4.6. □

References

[1] C. Kimberling, Interspersions and dispersions, *Proceedings of the American Mathematical Society* **117**(2) (1993), 313–321.
[2] C. Kimberling, The first column of an interspersion, *Fibonacci Quarterly* **32**(4) (1994), 301–315.
[3] C. Kimberling, Complementary equations, *Journal of Integer Sequences* **10**(1) (2007), 07.1.4.
[4] C. Kimberling, Complementary equations and Wythoff sequences, *Journal of Integer Sequences* **11**(3) (2008), 08.3.3.
[5] J. Parker Shectman, A quilt after Fibonacci, part 1 of 3: Construction of the quilt, (2021), (electronic), `https://arxiv.org/archive/math`
[6] J. Parker Shectman, A quilt after Fibonacci, part 2 of 3: Cohorts, free monoids, and numeration, (2021), (electronic), `https://arxiv.org/archive/math`
[7] N. J. A. Sloane, On-line encyclopedia of integer sequences, (1964-), (electronic), `http://oeis.org/`

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