A QUILT AFTER FIBONACCI, PART 1 OF 3: CONSTRUCTION OF THE QUILT

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Dedicated to Professor Saber Elaydi

Abstract. A *stair-cone* tiling fills the first quadrant of the plane with unit-square cells — white and black — while satisfying certain properties. The investigation begins with a stair-cone tiling that satisfies

Property 1. *The count of black cells in the i^{th} row equals i and the count of black cells in the j^{th} column equals j,*

together with other properties. This first part of the paper presents multiple constructions of this stair-cone and shows the equivalence of the constructions.

One construction evolves like a cellular automaton, another resembles a quilt. Having identical black and white regions, the stair-cone and quilt provide a graphical calculus for identities via the integer sequences they share. These sequences describe the stair-cone and quilt geometry, specifying integer coordinates for extrema of their regions.

To further study graphical and numerical sequences arising from the stair-cone and quilt geometry, the paper defines a *cohort*. Elaborating on the idea of cohort, Part 2 of the paper uses *cohort sequences* to produce convenient formulas for several types of integer sequences, including those which arise in the quilt and stair-cone.

The quilt's larger squares and rectangles partition the stair-cone sequences into subsequences, thus allowing them to be written as two-dimensional arrays. In Part 3 of the paper, these arrays turn out to be interspersion–dispersion arrays or to satisfy the relaxed definition of *interspersoid–dispersoid* array. In both cases, the paper characterizes the structure of blocks of rows with respect to interspersion, to reveal one more aspect of the quilt that exhibits a remarkable degree of self-similarity.

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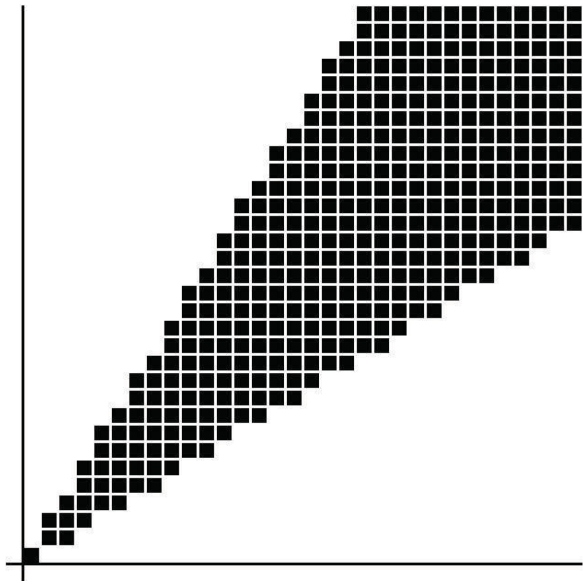

FIGURE 1. A finite corner patch of the stair-cone tiling

1. Introduction

This paper considers stair-cone tilings and quilt tilings and their use as a graphical calculus for identities between integer sequences. In particular, the regions of black and white of a stair-cone (Figure 1) are identical to the regions of black and white of the corresponding quilt (Figure 2). The constructions reference a unit-square grid, and reside in quadrant I of the plane, extending semi-infinitely in both directions (northward and eastward).

For the stair-cone, the border between its black and white regions fully describes its geometry. Coordinates of black cells that border white cells form integer sequences that describe the stair-cone in a concise way, since this border comprises comparatively fewer cells than the regions themselves. Consider the interval of black cells in each row of Figure 1, that is, the column indices of the first black cell and the last black cell in each row, *e.g.*, $[1,1], [2,3], [2,4], [3,6]$. The resulting sequences of column indices $1,2,2,3,4,4,5,5,6,7,\ldots$, respectively $1,3,4,6,8,9,11,12,14,\ldots$, describe all outside corners at the left, respectively, right borders of the black region.

Remark 1.1. The initiated reader will recognize these pairs as $\left[\lfloor n/\phi \rfloor, \lfloor n\phi \rfloor\right]$, $n = 1,2,\ldots$, sequences that have been studied for more than a century. In the context of this history, the geometric approach of the stair-cone tiling makes three contributions. First, the stair-cone offers an alternative pedagogical device: A way to arrive at these special sequences through direct geometric constructions (Methods 1 and 2), without introducing two-player games. Secondly, the axiomatic treatment of the stair-cone construction (Properties 1, 2, and 3), allows systematic variations and extensions (Example 4.1, Figure 4), which follow immediately from relaxing individual axioms. Lastly, the stair-cone being expressed as a tiling motivates its investigation jointly with that of another tiling — the quilt (Method 3), which shows the equivalence of the black and white regions of the two (Corollary 3.5).

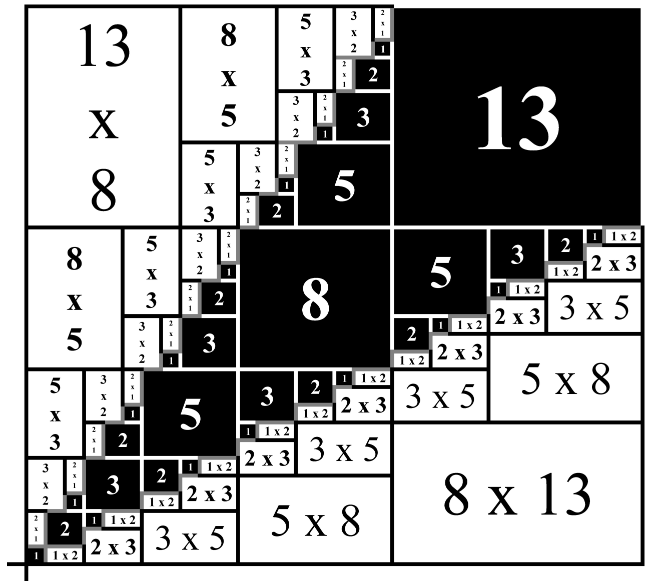

FIGURE 2. A finite corner patch of the quilt tiling

For the stair-cone, the evolution of its black region's border resembles a cellular automaton (Formula 13). By contrast, the quilt provides another method of generating the same border.

Turning our attention to the quilt (Figure 2), we find it repeats black squares and white rectangles of various sizes in an apparently self-similar arrangement. Just two *quilt indices* completely describe each quilt square, the (ordinal) size of the quilt square being one of that square's quilt indices, the other being its *age*, that is, its *birth order* with respect to the method that generates the quilt (Section 5.3). In each *generation*, that is, each action u of this method, a square already in the quilt can spawn only one new square — a square of the same size as its parent. Ultimately, the family of quilt squares of a given size stems from a single ancestor and can be fully ordered. Thus an integer pair comprising the size and birth order uniquely specifies the position of a square in the quilt. Note that these "quilt indices" differ from Cartesian row and column coordinates used to describe individual unit-square cells of the stair-cone.

Superimposing the quilt and stair-cone reveals that no two outside corners of the black stair-cone lie in the same quilt square. Thus the quilt's internal borders (the "grout" between same-colored tiles) partition the set of black outside-corner cells of the stair-cone. By means of this partition, the coordinate sequences of stair-cone corners fall into ordered collections of subsequences (Corollary 3.5). Part 3 of the paper [3] will show that in each case, the resulting collection of subsequences comprises — without reordering — either an interspersion, or a relaxed version, which the paper calls *interspersoid.*

The paper will also examine the integer sequences that describe the geometry of the quilt, beyond their connection to the stair-cone. To construct the quilt (Section 5.3), each action u of the method replicates a portion of the existing quilt, (the domain of action u). Because the domains of consecutive actions overlap, each action generally captures two prior generations of squares, causing the population of squares of each size to increase geometrically (Specifically, action u captures two generations of squares of sizes $1 \leq k \leq u-2$).

Thus, stated in terms of quilt squares, the populations of squares increase even though an individual square can spawn only one new child in each generation, since the *reproductive life* of a square is two generations, in general. To be precise, each square along the quilt's diagonal begets one child square, while each off-diagonal quilt squares begets two children — one in each of the next two generations.

The method of Section 5.3 also associates to each square in the quilt a *genealogy* — an integer tuple that records the sequence of generations u at which that square and each of its ancestors first appeared. Except for the squares along the main diagonal (all of whose genealogies are the empty tuple), genealogies are unique, allowing a total lexicographic ordering of the off-diagonal squares in the quilt.

For each size of square (1×1, 2×2, 3×3, 5×5,...), an examination of Figure 2 shows more 1×1's than 2×2's, more 2×2's than 3×3's, and so forth, in any finite corner patch of the quilt. In fact, the population of squares of size k lags the population of squares of size $k-1$ by one generation (Corollary 6.7). Part 2 of the paper [2] will show that the genealogy for the n^{th} quilt square of ordinal size $k = 2, 3, 4, \ldots$ equals $k-1$ plus the genealogy for the n^{th} quilt square of size one, and, moreover, that the genealogy for the n^{th} quilt square of size 1 equals a certain *maximal Fibonacci expansion* of n. In this context of Fibonacci numeration, Part 2 of the paper shows that the main quilt sequence $a_{n,k}$ (Table 1) constitutes a sort of dual to the Wythoff array.

As each action of the method adds new squares to the quilt, new integers — the coordinates of these new squares — are composed. The wording "composed" is deliberate, because the aforementioned Fibonacci expansions correlate quilt squares with *restricted compositions*, that is, ordered partitions of a positive integer. For example, as defined in the next section of the paper, let $a_{n,k}$ be southernmost row of the n^{th} quilt square of ordinal size k. Part 2 of the paper [2] will show that while integer $a_{n,k}$ encodes a composition of some integer $F^{-1}(n) + k - 1$, $a_{n,k+1}$ encodes a related composition of integer $F^{-1}(n) + k$. The compositions are those restricted to 1s and 2s and the encoding proceeds by taking partial sums of these 1's and 2's applying the partial sums as coefficients of the maximal Fibonacci expansion (Figure 10). Thus, the sequences of integers that describe the quilt geometry share the self-similar quality apparent when viewing the quilt.

2. Notation

General:	
$\phi \equiv (\sqrt{5}+1)/2$,	The Golden Ratio;
$F_{k+1} = F_k + F_{k-1}$, $k \geq 1$, with $F_0 = 0$ and $F_1 = 1$	The Fibonacci numbers;
<u>123456</u>,	An integer sequence from Sloane's OEIS [4];
<u>123456</u>$_n$,	The sequence reindexed (relative to the 'list' in OEIS);

The stair-cone (Figure 1):

$(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$,	Coordinate pair for cell in row i and column j;
$A(z)$,	Starts of rows or columns;
$\Omega(z)$,	Ends of rows or columns;
$C(z)$,	Count of black cells in rows or columns;
B, W,	Regions of black, respectively, white squares;
χ_B, χ_W,	Their respective indicator functions on $\mathbb{Z}_+\times \mathbb{Z}_+$;
χ^C,	An indicator induced by a count function $C(z)$;

The quilt (Figure 2):

$[a,b] \times [c,d]$,	A 2-D interval of rows $\times$ columns;
$[a,b] \times [c,d] + r \times s \equiv$ $[a+r, b+r] \times [c+s, d+s]$,	Typical interval arithmetic (scalar addition);

Black squares in the quilt (Tables 1, 2, 3, and 4):

$S_{0,k} = [a_{0,k}, b_{0,k}] \times [c_{0,k}, d_{0,k}] \subset \mathbb{Z}_+\times \mathbb{Z}_+$	A Black quilt square lying on the main diagonal;
$S_{n,k} \subset \mathbb{Z}_+\times \mathbb{Z}_+$,	A pair of equivalent squares: $[a_{n,k}, b_{n,k}] \times [c_{n,k}, d_{n,k}]$ below the diagonal, and $[c_{n,k}, d_{n,k}] \times [a_{n,k}, b_{n,k}]$ above the diagonal;

White rectangles in the quilt:

$R_{n,k} \subset \mathbb{Z}_+\times \mathbb{Z}_+$,	A pair of equivalent rectangles: $[\alpha_{n,k}, \beta_{n,k}] \times [\gamma_{n,k}, \delta_{n,k}]$ below the diagonal, and $[\gamma_{n,k}, \delta_{n,k}] \times [\alpha_{n,k}, \beta_{n,k}]$ above the diagonal;

Genealogy and Cohorts:

$S_1, S_2, \ldots,$	A sequence of integers;
$S_1 S_2 \cdots,$	The equivalent integer word;
$C_1, C_2, \ldots, C_t, \ldots,$	A sequence of Cohorts;
$C_{0,k}, C_{1,k}, \ldots, C_{t,k}, \ldots,$	Cohorts of quilt squares of *ordinal size* k (cardinal size $F_{k+1} \times F_{k+1}$);
$D_{1,k}, D_{2,k}, \ldots, D_{t,k}, \ldots,$	Cohorts of quilt rectangles of *ordinal size* k (cardinal size $F_{k+1} \times F_{k+2}$);
$\oplus$,	Concatenation of tuples;
$v_{n,k}$,	Genealogy of a quilt square.

3. Main Results

3.1. A stair-cone after Fibonacci.

Proposition 3.1. *For $n = 1, 2, \ldots$, row (column) n of Figure 1 begins in column (row) $A(n) = \lceil n/\phi \rceil$* 019446 *and ends in column (row) $\Omega(n) = \lfloor n\phi \rfloor$* 000210*. Thus, the methods for constructing the Figure are also methods for calculating these spectrum sequences.*

Proof. Follows from Proposition 6.1 plus Proposition 4.2 (strong construction) for Method 1, or from Proposition 6.2 (weak construction) for Method 2. □

3.2. **A Quilt after Fibonacci.** For the planar construction in Figure 2, let "north," "south," "east," and "west" indicate the usual compass directions, and consider its black squares. For $k \geq 1$, squares $S_{0,k} = [a_{0,k}, b_{0,k}] \times [c_{0,k}, d_{0,k}]$ lie on the main diagonal, *i.e.*, $a_{0,k} = c_{0,k}$, $b_{0,k} = d_{0,k}$. Moreover, square $S_{0,k}$ covers $F_{k+1} \times F_{k+1}$ cells, *i.e.*, $b_{0,k} - a_{0,k} + 1 = d_{0,k} - c_{0,k} + 1 = F_{k+1}$. Thus, the index k gives the *ordinal* size of the square.

1	2	4	7	12	20	33	54
3	5	9	15	25	41	67	109
6	10	17	28	46	75	122	198
8	13	22	36	59	96	156	253
11	18	30	49	80	130	211	342
14	23	38	62	101	164	266	431
16	26	43	70	114	185	300	486
19	31	51	83	135	219	355	575

Table 1. Table of $a_{n,k}$, for $n = 0, 1, \ldots, 7$ and $k = 1, 2 \ldots, 8$

1	3	6	11	19	32	53	87
3	6	11	19	32	53	87	142
6	11	19	32	53	87	142	231
8	14	24	40	66	108	176	286
11	19	32	53	87	142	231	375
14	24	40	66	108	176	286	464
16	27	45	74	121	197	320	519
19	32	53	87	142	231	375	608

Table 2. Table of $b_{n,k}$, for $n = 0, 1, \ldots, 7$ and $k = 1, 2 \ldots, 8$

For $n > 0$, $S_{n,k}$ represents the square $[a_{n,k}, b_{n,k}] \times [c_{n,k}, d_{n,k}]$ below the diagonal and its mirror above the diagonal, $[c_{n,k}, d_{n,k}] \times [a_{n,k}, b_{n,k}]$. For each k, count the pairs of squares $S_{1,k}, S_{2,k}, \ldots,$ of the k^{th}-smallest size from southwest to northeast, with $n = 1$ indicating the first pair, $n = 2$ indicating the second, etc. Tables 1 through 4 tabulate the first few elements of $a_{n,k}$, $b_{n,k}$, $c_{n,k}$ and $d_{n,k}$.

For $k \geq 1$, $R_{n,k}$ represents the rectangle $[\alpha_{1,k}, \beta_{1,k}] \times [\gamma_{1,k}, \delta_{1,k}]$ below the diagonal and its mirror above the diagonal, $[\gamma_{n,k}, \delta_{n,k}] \times [\alpha_{n,k}, \beta_{n,k}]$. The rectangle

1	2	4	7	12	20	33	54
4	7	12	20	33	54	88	143
9	15	25	41	67	109	177	287
12	20	33	54	88	143	232	376
17	28	46	75	122	198	321	520
22	36	59	96	156	253	410	664
25	41	67	109	177	287	465	753
30	49	80	130	211	342	554	897

TABLE 3. Table of $c_{n,k}$, for $n = 0, 1, \ldots, 7$ and $k = 1, 2 \ldots, 8$

1	3	6	11	19	32	53	87
4	8	14	24	40	66	108	176
9	16	27	45	74	121	197	320
12	21	35	58	95	155	252	409
17	29	48	79	129	210	341	553
22	37	61	100	163	265	430	697
25	42	69	113	184	299	485	786
30	50	82	134	218	354	574	930

TABLE 4. Table of $d_{n,k}$, for $n = 0, 1, \ldots, 7$ and $k = 1, 2 \ldots, 8$

below the diagonal covers $F_{k+1} \times F_{k+2}$ cells and its counterpart above the diagonal covers $F_{k+2} \times F_{k+1}$ cells, *i.e.*, $\beta_{n,k} - \alpha_{n,k} + 1 = F_{k+1}$ and $\delta_{n,k} - \gamma_{n,k} + 1 = F_{k+2}$. In this way, the index k gives the *ordinal* size of the rectangle, the ratio of height to width being fixed at F_{k+1}/F_{k+2} for each k.

For $k \geq 1$, counting the pairs of rectangles $R_{1,k}, R_{2,k}, \ldots,$ of the k^{th}-smallest (ordinal) size from southwest to northeast, $n = 1$ indicates the first pair, $n = 2$ indicates the second, etc. In particular, the pair of rectangles $R_{1,k}$ comprises one rectangle lying against the east–west (horizontal) axis, *i.e.*, $\alpha_{1,k} = 1$, and a second rectangle lying against the north–south (vertical) axis, *i.e.*, $\gamma_{1,k} = 1$.

Proposition 3.2 (Cohort-based formulas). *For the black squares* $S_{n,k} = [a_{n,k}, b_{n,k}] \times [c_{n,k}, d_{n,k}]$ *in Figure 2,* $n = 0, 1, 2, \ldots,$ $k = 1, 2, 3 \ldots,$

$$a_{n,k} = F_{k+2} + \lfloor n\phi \rfloor F_{k+1} + nF_k - 1; \tag{1}$$

$$b_{n,k} = F_{k+3} + \lfloor n\phi \rfloor F_{k+1} + nF_k - 2; \tag{2}$$

$$c_{n,k} = F_{k+2} + \lfloor n\phi \rfloor F_{k+2} + nF_{k+1} - 1; \tag{3}$$

$$d_{n,k} = F_{k+3} + \lfloor n\phi \rfloor F_{k+2} + nF_{k+1} - 2; \tag{4}$$

(Tables 1, 2, 3, and 4), whereas for the white rectangles $R_{n,k} = [\alpha_{n,k}, \beta_{n,k}] \times [\gamma_{n,k}, \delta_{n,k}]$ *in Figure 2,* $n = 1, 2, 3 \ldots,$ $k = 1, 2, 3 \ldots,$

$$\alpha_{n,k} = -F_{k+1} + \lfloor n\phi \rfloor F_k + nF_{k-1} + 1; \tag{5}$$

$$\beta_{n,k} = \lfloor n\phi \rfloor F_k + nF_{k-1}; \tag{6}$$

$$\gamma_{n,k} = F_{k+1} + \lfloor n\phi \rfloor F_{k+1} + nF_k - 1; \tag{7}$$

$$\delta_{n,k} = F_{k+3} + \lfloor n\phi \rfloor F_{k+1} + nF_k - 2. \tag{8}$$

Proof. In Part 2 of this paper [2]. □

The following Corollary of Propositions 3.1 and 3.2, identifies versions of the Fibonacci word in the sequence $\Omega_n - \Omega_{n-1}$ of steep-staircase risers or gentle-staircase treads, and in the spacings between like-sized quilt squares and like-sized quilt rectangles.

Corollary 3.3 (Fibonacci Word in the Quilt and stair-cone)**.** $\forall n, k \in \mathbb{Z}^+$

$$\begin{aligned}\underline{\texttt{005614}}(n-2) &= A_n - A_{n-1},\\ &= \Omega_n - \Omega_{n-1} - 1,\\ &= (\alpha_{n,k} - \alpha_{n-1,k} - F_{k+1})/F_k,\\ &= (\beta_{n,k} - \beta_{n-1,k} - F_{k+1})/F_k,\\ &= (a_{n,k} - a_{n-1,k} - F_{k+2})/F_{k+1},\\ &= (b_{n,k} - b_{n-1,k} - F_{k+2})/F_{k+1},\\ &= (\gamma_{n,k} - \gamma_{n-1,k} - F_{k+2})/F_{k+1},\\ &= (\delta_{n,k} - \delta_{n-1,k} - F_{k+2})/F_{k+1},\\ &= (c_{n,k} - c_{n-1,k} - F_{k+3})/F_{k+2},\\ &= (d_{n,k} - d_{n-1,k} - F_{k+3})/F_{k+2}.\end{aligned}$$

Proof. Direct calculation from Proposition 3.1 and (1)–(8). □

Proposition 3.4 (Spectrum relationship between a and d)**.**

$$d_{n,k} = \lfloor a_{n,k}\phi \rfloor, n = 0, 1, 2, \ldots, k = 1, 2, 3 \ldots. \tag{9}$$

Proof. In Part 3 of this paper [3]. □

Corollary 3.5 (Equivalence of Figures 1 and 2)**.** *Consider the rightmost black cells of all rows i of Figure 1. Their coordinates (i, Ω_i) can be written as the union of the coordinates $(a_{n,k}, d_{n,k})$ for the southeastern corners of all black squares $S_{n,k}$ on or below the diagonal in Figure 2, or equivalently, the northwestern corners of all black squares on or above the diagonal. That is, $\{(i, \Omega_i)\}_{i=1}^{\infty} = \{(a_{n,k}, d_{n,k})\}_{n=0,k=1}^{\infty,\ \infty}$.*

Proof. In Part 3 of the paper [3]. □

4. Stair-cone & Quilt Preliminaries

4.1. **The canonical induced stair.** First, establish notation to describe the stair-cone in Figure 1. In the succeeding, coordinates of the pair (i, j) indicate a row counted from south to north, followed by a column counted from west to east. Note that these coordinates are ordered vertical-first, rather than the usual horizontal-first order for rectilinear coordinates.

Regions of the stair-cone: Figure 1 partitions $\mathbb{Z}_+ \times \mathbb{Z}_+$ into regions of black cells and white cells,

$$\begin{aligned}B &= \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid \text{black cell in row } i \text{ in column } j\}, \text{ and}\\ W &= \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid \text{white cell in row } i \text{ in column } j\}, \text{ with}\\ B &\cup W = \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ and } B \cap W = \emptyset.\end{aligned}$$

Indicators of the regions: Associated with this partition are χ_B and χ_W, the (0-1-binary-valued) indicator functions of B and W, respectively. That is,

$$\chi_B(i,j) = \begin{cases} 1, & (i,j) \in B; \\ 0, & (i,j) \notin B; \end{cases} \quad \text{and} \quad \chi_W(i,j) = \begin{cases} 1, & (i,j) \in W; \\ 0, & (i,j) \notin W. \end{cases}$$

Property. $1'$ *(Stair induced from a (univariate) count)* To generalize the row and column *count* of Property 1, let $C(n)$ denote a positive-integer-valued function over the positive integers. Then over pairs of positive integers (i,j), $C(n)$ may *induce* corresponding 0-1-binary-valued function(s) χ^C satisfying the relation $\sum_i \chi^C(i,n) = \sum_j \chi^C(n,j) = C(n)$.

Consider the identity function I defined by $I(n) = n$. Then the indicator χ_B, of the region B, is, in fact, the canonical representative (Definition 4.2) of the χ^I, being the particular χ^I that satisfies additional Properties 2 and 3 described next.

Of constructions that satisfy Property 1, the particular construction shown in Figure 1 has three regions. The region of black cells includes the diagonal, has no inclusions of white cells, and isolates two regions of white cells, each having no inclusions of black cells. Formally label these two subregions of W

$$W^{\text{lower}} = \big\{(i,j) \in W \,|(k,j) \in W, \forall k = 1,\ldots,i \text{ and } (i,h) \in W, \forall h > j\big\} \text{ and}$$
$$W^{\text{upper}} = \big\{(i,j) \in W \,|(i,h) \in W, \forall h = 1,\ldots,j \text{ and } (k,j) \in W, \forall k > i\big\}.$$

In Figure 1, the individual regions W^{lower}, W^{upper}, and B, satisfy Property 2:

Property 2 (Row–Column convexity, abbreviated RC convexity). For region $R \subset \mathbb{Z}_+ \times \mathbb{Z}_+$, call R and its characteristic function χ_R *row convex*, respectively, *column-convex*, when satisfying the following subproperties:

Row convexity: If $(i,j) \in R$ and $(i,j-1) \notin R$, then $(i,h) \notin R$ for all $h = 1,\ldots,j-1$, and if $(i,j) \in R$ and $(i,j+1) \notin R$, then $(i,h) \notin R$ for all $h = j+1, j+2, \ldots$.

Column convexity: If $(i,j) \in R$ and $(i-1,j) \notin R$, then $(k,j) \notin R$ for all $k = 1,\ldots,i-1$, and if $(i,j) \in R$ and $(i+1,j) \notin R$, then $(k,j) \notin R$ for all $k = i+1, i+2, \ldots$.

Remark 4.1. Row convexity of a region R allows the definition of unique row starts $A_R^{\text{row}}(i) = \min_{(i,j)\in R} j$ and row ends $\Omega_R^{\text{row}}(i) = \sup_{(i,j)\in R} j$, respectively. Likewise, column convexity allows the definition of unique column starts $A_R^{\text{col}}(j) = \min_{(i,j)\in R} i$ and column ends $\Omega_R^{\text{col}}(j) = \sup_{(i,j)\in R} i$.

Here, the supremum is taken over $\mathbb{Z}_+ \cup \{\infty\}$, so that, for example, $\Omega_{W^{\text{lower}}}^{\text{row}}(i) = \infty$ for all $i \geq 1$ and $\Omega_{W^{\text{upper}}}^{\text{col}}(j) = \infty$ for all $j \geq 1$, whereas, for the bounded region, we have $\Omega_B^{\text{row}}(i) = \max_{(i,j)\in B} j$ and $\Omega_R^{\text{col}}(j) = \max_{(i,j)\in R} i$.

Definition 4.1 (Partial Completion of the Constructions)**.** For R an induced stair, let R_n indicate a the partial construction of R with rows completed through row n and columns completed through column n, that is, $R_n {=} R \cap ([1, \Omega_R^{\text{row}}(n)] \times [1,\infty] \cup [1,\infty] \times [1, \Omega_R^{\text{col}}(n)])$. This notation allows the following definition of the third and last property satisfied by a canonical induced stair.

Property 3 (Diagonal Containment). Designate $R \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ and its characteristic function χ_R *diagonal containing* when $(n,n) \in R_n, \forall n \in \mathbb{Z}_+$.

Observe that region B of Figure 1 indeed satisfies Property 3. Lemma 4.1 shows a general result for canonical induced stairs.

Lemma 4.1. *Let $C(n)$ on $\mathbb{Z}_+$ induce χ^C on $\mathbb{Z}_+ \times \mathbb{Z}_+$, as described by Property 1′. Further, let χ^C indicate a region $R \subset \mathbb{Z}_+ \times \mathbb{Z}_+$, that is, $\chi_R \equiv \chi^C$. Now, suppose that χ^C and R satisfy Properties 2 and 3. Then (i) χ^C is the unique indicator induced by C, and R is the region it indicates and (ii) $\chi^C(i,j)$ is symmetric in its arguments i and j, in particular, $\forall n \geq 1$, $A_R^{row}(n) = A_R^{col}(n) = A_R(n)$ and $\Omega_R^{row}(n) = \Omega_R^{col}(n) = \Omega_R(n)$.*

Proof (Induction on n). Consider the partial completions of R, per Definition 4.1. For $n = 1$, since R satisfies Property 3, R_1 is uniquely defined as the symmetric region $[1,1]\times[1,C(1)]\cup[1,C(1)]\times[1,1] = \{(1,1),...,(1,C(1))\}\cup\{(1,1),...,(C(1),1)\}$.

Suppose R_n satisfies (i) and (ii). By Property 3, $(n+1,n+1) \in R_{n+1}$. By Property 2, therefore, $A_R^{\text{row}}(n+1) \leq n+1$, and row $n+1$ is uniquely defined as the cells $(n+1, A_R^{\text{row}}(n+1)), \dots, (n+1, A_R^{\text{row}}(n+1) + C(n+1) - 1)$. The hypothesis of symmetry for R_n ensures that $(A_R^{\text{row}}(n+1), n+1), \dots, (n, n+1) \in R_n$ and $(1, n+1), \dots, (A_R^{\text{row}}(n+1) - 1, n+1) \notin R_n$. Thus, column $n+1$ of R_{n+1} is uniquely defined as $(A_R(n+1), n+1), \dots, (A_R(n+1)+C(n+1)-1, n+1)$, making R_{n+1} symmetric, with $A_R^{\text{row}}(n+1) = A_R^{\text{col}}(n+1) = A^R(n+1)$ and $\Omega_R^{\text{row}}(n+1) = \Omega_R^{\text{col}}(n+1) = \Omega_R(n+1) = A_R(n+1) + C(n+1) - 1$. □

Definition 4.2. If, as in Lemma 4.1, χ^C of Property 1′ also satisfies Properties 2 and 3, designate the set of cells indicated by χ^C as the *canonical stair induced by* C. In Figure 1, B uniquely satisfies Properties 1, 2, and 3. Accordingly, B is the canonical stair-cone induced by $C = I$, and can be characterized as follows.

Proposition 4.2. *B satisfying Properties 1, 2, and 3 implies that $A_B(n) = \lceil n/\phi \rceil$ and $\Omega_B(n) = \lfloor n\phi \rfloor$.*

Proof. First, show that $A(\Omega(n)) = n$, $\forall n \in \mathbb{Z}_+$.

Using Property 2, reformulate Property 3 as $A_B(i) \leq i \leq \Omega_B(i)$, $\forall i \in \mathbb{Z}_+$.

Considering i as a row index, Property 2 implies that the zone of black cells in column $A(i)$ must be contiguous with cell $(i, A(i))$. That is, $(i, A(i)), \dots, (i, \Omega(i)) \in B$. By Property 3, $(A(i), A(i)) \in B$, hence by Property 2, $(A(i), A(i)), \dots, (i, A(i)) \in B$. By symmetry (Lemma 4.1*(ii)*), $(A(i), A(i)), \dots, (A(i), i) \in B$, in particular, $(A(i), i-1) \in B$.

Now suppose that $(\Omega(i), i-1) \in B$. By Property 2, this would imply that $(A(i), i-1), \dots, (\Omega(i), i-1) \in B$, a total of i cells (at least) in column $i-1$. Thus, $\Omega(i-1) - A(i-1) + 1 \geq i$, contradicting Property 1. Therefore, $(\Omega(i), i-1) \notin B$, and because $(\Omega(i), i) \in B$, it must be that $A(\Omega(i)) = i$.

Thus, $\forall\, n \in \mathbb{Z}_+$, $\exists A^{-1}(n) \in \mathbb{Z}_+$ with $A^{-1}(n) = \Omega(n) \geq n$. Substituting into $\Omega(n) = A(n) + n - 1$ (Property 1) gives

$$A^{-1}(n) - n + 1 = A(n), \tag{10}$$

for which $A(n) = \lceil n/\phi \rceil$, $\Omega(n) = A^{-1}(n) = \lfloor n\phi \rfloor$, is the only solution satisfying Property 3, with $A(n) \leq n \leq A^{-1}(n)$, $\forall n \in \mathbb{Z}_+$. □

Example 4.1. The equation (10) involving A and its (functional) inverse is an analog — for integer-valued functions over $\mathbb{Z}_+$ — of the equation $\phi - 1 = \phi^{-1}$ defining the golden ratio ϕ in relation to its (multiplicative) inverse (see Figure 3). Solutions to (10) not satisfying Property 3 exist over bounded portions of $\mathbb{Z}_+$. For example, $A(n) = N + 1 - n$ and $A^{-1}(n) \equiv N$ over $1 \leq n \leq N$, for fixed $N \in \mathbb{Z}_+$.

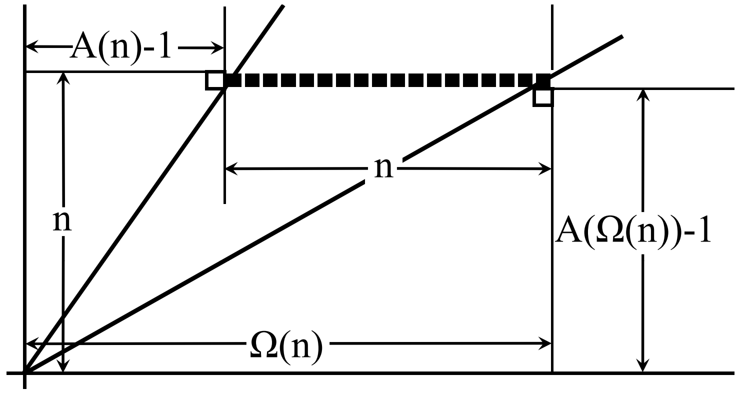

FIGURE 3. Stair-cone as a Discrete Analog of the Golden Ratio Formula

4.2. **A Stair-cone and Quilt after Pell.**

Definition 4.3 (Stair-Cone)**.** When the count function of $C(n)$ in Property 1′ is a multiple of n, call the induced staircase R such that $\chi_C \equiv \chi^R$ a *stair-cone.*

Consider doubling the row and column count of Property 1. That is, let $C(n) = 2n$ in Property 1′, and designate the resulting stair-cone the *Pell stair-cone.* Next, consider a sequence of squares along its diagonal, where the northeast corner of a square touches the southwest corner of the subsequent one without overlapping. The size of the spinal squares follow the Pell sequence $P_n = 1, 2, 5, 12, 29, 70, \ldots$. That is, they have sizes 2 x 2, 5 x 5, 12 x 12, 29 x 29, and so forth.

The remainder of this stair-cone can be "quilted" using a combination of squares of sizes $P(n) \times P(n)$ and rectangles $P(n) \times P(n+1)$, for $n \geq 1$ (Figure 4).

While the remainder of the paper focuses on the stair-cone and quilt after Fibonacci, the Pell and other quilts remain objects of ongoing study [1]. Note that unlike the Pell stair-cone, the stair-cone after Fibonacci (black region of Figure 2) can be quilted in a *fully self-similar* fashion, using only squares from its own spine.

5. Construction Methods

This section presents two methods to construct the stair-cone (Figure 1), the strong method (Method 1) and the weak method (Method 2), as well as a method, Method 3, to construct the quilt (Figure 2). Method 1 and 3 have a black version and a white version, which directly define the black and white regions, respectively, with the other region then defined as its complement in Quadrant I.

5.1. **Strong methods to construct the stair-cone.**

Method 0 (Direct Method).

$$\begin{aligned} \chi_B : \quad \mathbb{Z}_+ \times \mathbb{Z}_+ \quad &\to \quad \{0, 1\} \\ (i, j) \quad &\mapsto \quad \begin{cases} 1, & {}^1\!/\!\phi < \dfrac{i}{j} < \phi; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

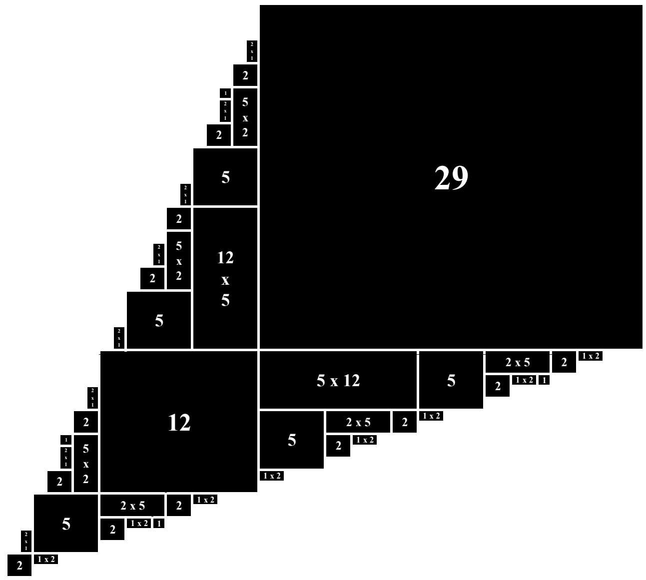

FIGURE 4. A Partial Quilting of the Pell stair-cone

If we knew Proposition 3.1 *a priori*, namely that the corners of outer squares in the stair-cone tiling touched the cone with irrational slopes $1/\phi$ and ϕ, then we could use Method 0.

Method 1 (Strong Method).

Black Version:
Start with a white canvas having a single black cell at the corner position $(1, 1)$. Propagate the construction outward considering successive antidiagonals $\{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ | i + j = n\}$, $n = 3, 4, \ldots$ one at a time. Blacken a cell at position (i, j) if row i has fewer than i black cells *and* column j has fewer than j black cells. Formally, the method describes the indicator function χ_B of B via:

$$\begin{aligned} \chi_B : \quad \mathbb{Z}_+ \times \mathbb{Z}_+ \quad &\to \quad \{0,1\} \\ (i,j) \quad &\mapsto \begin{cases} 1, & \sum_{h=1}^{j-1} \chi_B(i,h) < i \text{ and } \sum_{k=1}^{i-1} \chi_B(k,j) < j; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{11}$$

White Version:
Start with a black canvas and consider that the corner cell will not turn white, that is, $(1, 1) \notin W$. Propagate the construction outward considering successive antidiagonals $\{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \, | i + j = n\}$, $n = 3, 4, \ldots$ one at a time. Whiten a cell at position (i, j) if row i has fewer than $i - j$ white cells *or* column j has fewer than $j - i$ white cells. Formally, the method describes the indicator function χ_W of W via:

$$\begin{aligned} \chi_W : \mathbb{Z}_+ \times \mathbb{Z}_+ \quad &\to \quad \{0,1\} \\ (i,j) \quad &\mapsto \begin{cases} 1, & \sum_{h=1}^{j-1} \chi_W(i,h) < j - i \text{ or } \sum_{k=1}^{i-1} \chi_W(k,j) < i - j; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{12}$$

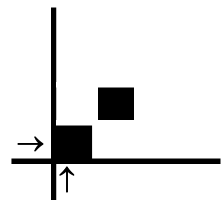

Figure 5. Initial seed pattern for Method 2

Remark 5.1. The strong method (Method 1) provides a more direct implementation of Property 1 as a formula than the weak method (Method 2). Since the recursions start with $(1,1) \in B$, respectively $(1,1) \notin W$, and propagate outward along successive antidiagonals $i + j = 3, 4, \ldots$, either version produces the canonical figure with its three characteristic regions: white along the two axes and black along the diagonal, each color with no inclusions of the other. In the calculations for Method 1, however, the sums tested by the formula involve a number of summands that increases at each step, a shortcoming absent from Method 2, presented next.

5.2. **Weak method to construct the stair-cone.** At each step, the weak method extends a row (or column) beyond the end of the previous row (column) by either one or two cells. To determine the length of this extension, the weak method compares just two cells from earlier steps.

Thus, Method 2 is more analogous to a cellular automaton than Method 1, in that Method 1 must count all cells back to row 1 and column 1, whereas Method 2 compares only two prior cells (i, A_i) and $(i+1, A_{i+1})$ (13). On the other hand, in Method 2 the pair of completed cells that determine the length of current extension lag farther and farther behind at each step, not a typical reduced form for a cellular automaton or linear recurrence. One can also think of Method 2 as a telescoping version of Method 1, one whose recurrence has collapsed to just two terms.

Construction by Method 2 begins at the corner with an initial seed pattern $B_2 = \{(1,1), (2,2)\}$ of two black cells. At each stage of the method, track the row and the column most recently completed by marking their positions with arrows along the axes. For the initial pattern, designate the first row and first column as the most recently completed row and column, respectively, indicated by arrows in Figure 5.

At any given stage, the method completes a row or column in which blackening started — but was not completed — at a prior stage. If all rows with black cells are complete, the method must operate on a column. Likewise, if all columns with black cells are complete, the method must operate on a row. Often, the method allows a choice of whether to extend a row or to extend a column. Consequently, Method 2 allows a plurality of construction sequences (Figure 6). In any construction sequence, though, Method 2 only extends B within the row immediately above the most recently completed row *or* within the column to the immediate right of the most recently completed column.

Without loss of generality, suppose that at the present stage we are constrained to — or have chosen to — operate on a row. If the row last completed was row i, then the candidate row must be row $i+1$. Let $A_i = \min_{(i,j)\in B} j$ and $\Omega_i = \max_{(i,j)\in B} j$ denote the column indices of the leftmost and rightmost black cells of row i, respectively.

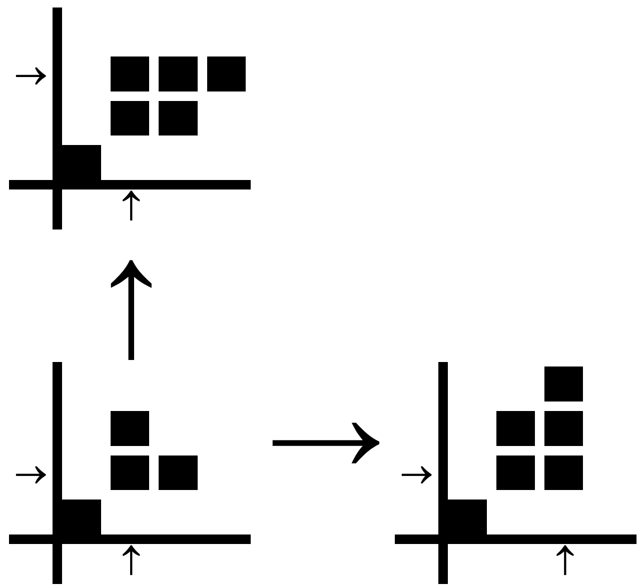

FIGURE 6. Alternatives for the Third Action of Method 2

Method 2 (Weak method). From Figure 5, observe that the initial pattern fixes $A_1 = 1$, $A_2 = 2$, and $\Omega_1 = 1$. Since we began blackening cells in candidate row $i + 1$ at a prior stage — but did not finish — we already know A_{i+1} but have not established Ω_{i+1} yet. We now complete row $i + 1$ by blackening cells through column

$$\Omega_{i+1} = \begin{cases} \Omega_i + 1, & A_{i+1} = A_i; \\ \Omega_i + 2, & A_{i+1} = A_i + 1; \end{cases} \tag{13}$$

letting cells in subsequent positions $(i + 1, \Omega_{i+1} + 1)$, $(i + 1, \Omega_{i+1} + 2), \ldots$ of row $i + 1$ remain white, and advancing the tracking arrow from row i to row $i + 1$ to mark it as the most recently completed row. Follow the symmetric analog of this process when completing a column rather than a row. For example, Figure 6 shows the possible operations at stage three of the construction based on the application of (13).

Beginning with the initial seed pattern (Figure 5), the method completes each previously started row (column) n to generate the row (column) ends Ω_n. New row (column) starts, A_n, automatically form in the process. This allows the transformation of (13) into a bivariate recurrence in A_n and Ω_n, as follows.

Without loss of generality, let row $n + 1$ start at $(n + 1, A_{n+1})$. Now, column A_{n+1} must end at $(\Omega_{A_{n+1}}, A_{n+1})$. Thus, $\Omega_{A_{n+1}} \geq n + 1$. By the dichotomy in (13), moreover, the only choices of A_{n+1} are $A_{n+1} = A_n$ and $A_{n+1} = A_n + 1$, and it may also be the case that $\Omega_{A_n} = n + 1$. Thus we may write:

$$\begin{aligned} A_1 &= \Omega_1 = 1; \\ A_{n+1} &= \operatorname{argmin}_{m \in \{A_n, A_n+1\}} \{\Omega_m \,|\, \Omega_m \geq n + 1\}, \quad n \geq 1; \\ \Omega_{n+1} &= \Omega_n + A_{n+1} - A_n + 1, \qquad\qquad n \geq 1. \end{aligned} \tag{14}$$

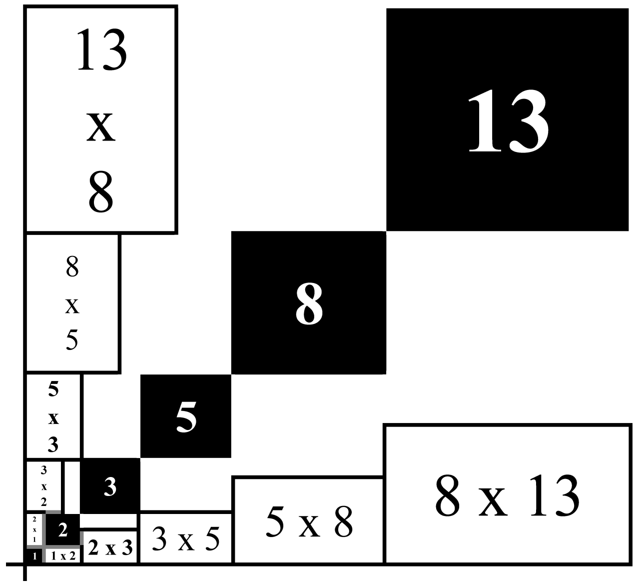

FIGURE 7. Black Spinal Squares and White Spinal Rectangles

Examining (14) for the dichotomy on Ω_{A_n}, we find that

$$A_{n+1} = \begin{cases} A_n, & \Omega_{A_n} = n+1; \\ A_n + 1, & \Omega_{A_n} = n; \end{cases}$$

thus allowing us to write a bivariate recurrence in A_n and Ω_n,

$$\begin{array}{lll} A_1 & = \Omega_1 = 1; & \\ A_{n+1} & = A_n + n + 1 - \Omega_{A_n}, & n \geq 1; \\ \Omega_{n+1} & = \Omega_n + A_{n+1} - A_n + 1, & n \geq 1. \end{array}$$

Section 6.2 will continue this development.

5.3. **Method of constructing the quilt.**

Method 3 (Quilt method).

Black Region:
Figure 7 shows a sequence of squares $k = 1, 2, \ldots$ of size F_{k+1} and with southwest cell at position $(F_{k+2} - 1, F_{k+2} - 1)$. Observe that the northeast corner of a square touches the southwest corner of the subsequent one without overlapping. Designate this particular sequence the *sequence of spinal squares*, since it is contained in B and B can be covered by taking this sequence as a spine and expanding it with additional squares according to the following procedure:

For $k = 1, 2, \ldots$, consider the block composed of all the black squares east of spinal square $k - 1$ and south of spinal square $k + 1$. In particular, this block comprises spinal square k together with all black squares to its south and to its east. Paste a clone of this block immediately east of spinal square $k + 1$, immediately south of spinal square $k + 2$, such that the outside northwest corner of the cloned block coincides with the inside corner formed where spinal squares $k + 1$ and $k + 2$

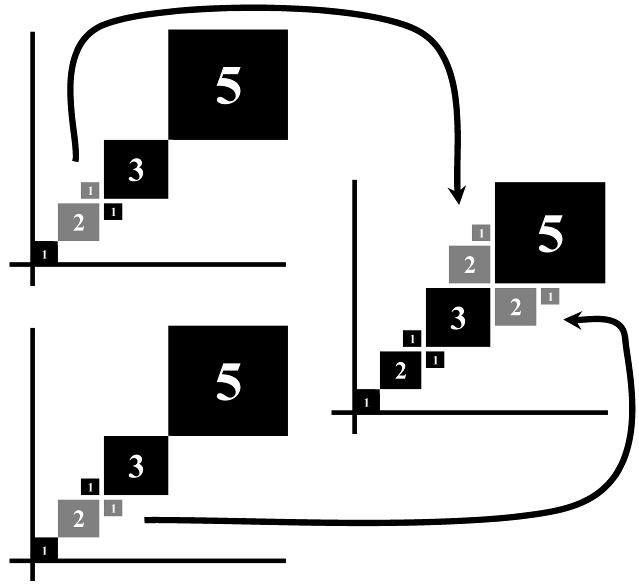

FIGURE 8. Second Action of Method 3 to Construct the Black Region

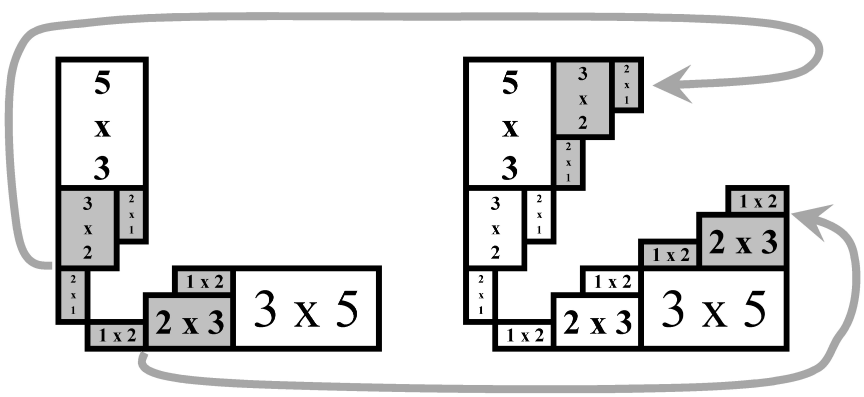

FIGURE 9. Second Action of Method 3 to Construct the White Region

touch. At each step, apply the symmetric analog of this process above the spine (Figure 8).

White Region:
Consider the white region W^{lower} southeast of B separately from the white region W^{upper} northwest of B. For W^{lower}, Figure 7 shows a sequence of rectangles beginning with a rectangle of dimension 1×2 placed with its western cell at position $(1, 2)$ and continuing eastward with rectangles placed along the horizontal axis, the k^{th} rectangle in this sequence having dimensions (rows $\times$ columns) $F_{k+1} \times F_{k+2}$, and its southwest cell at position $(1, F_{k+3} - 1)$. Designate this sequence, and its

image mirrored in the diagonal, as *sequences of spinal rectangles*, since not only does W contain these sequences, but taking these sequences as a spine, W may be covered by appending additional disjoint rectangles according to the following procedure:

Consider the lower sequence of spinal rectangles. Paste a clone of the first spinal rectangle to the immediate north of the second spinal rectangle, with the eastern edge of the clone aligned with the eastern edge of the spinal rectangle below it. Then for $k = 3, 4, \ldots$ successively, consider spinal rectangles of ordinal sizes $k-2$ and $k-1$ (cardinal sizes $F_{k-1} \times F_k$, respectively, $F_k \times F_{k+1}$), together with all white rectangles north of these spinal rectangles (yet south of B). Paste a clone of this block to the immediate north of the k^{th} spinal rectangle, with the eastern edge of the clone aligned with the eastern edge of the spinal rectangle below it. Also apply the symmetric analog of this process to W^{upper} (Figure 9).

6. Analysis of Construction Methods

6.0. **Investigation of Method 0.** Trivially, the stair produced by Method 0 is *diagonal-containing*, whereas $1/\phi < 1 < \phi$ and *RC-convex* by the monotonicity of the quotient on $\mathbb{Z}_+ \times \mathbb{Z}_+$. Moreover, Proposition 3.1 follows directly, since without loss of generality, $i = \lfloor j\phi \rfloor$ is the largest i for which $i/j < \phi$, whereas $i = \lceil j/\phi \rceil$ is the smallest i for which $i/j > 1/\phi$. It remains to show that the other methods are equivalent to Method 0.

6.1. **Investigation of Method 1.** This investigation first generalizes Method 1 by relaxing the row (column) count of (11) and (12) to values $C(i)$ and $C(j)$, as in Property 1′, but requiring C to be nondecreasing. The result of the relaxed method turns out to be an *RC-convex*, *diagonal-containing* induced stair (Properties 2 and 3, respectively).

Proposition 6.1. *Let $C(n)$ on $\mathbb{Z}_+$ induce χ^C on $\mathbb{Z}_+ \times \mathbb{Z}_+$, per Property 1′. Further, let this indicator χ^C partition $\mathbb{Z}_+ \times \mathbb{Z}_+$ into a set of black cells $R^C = \big\{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \big| \chi^C(i,j) = 1\big\}$ and white cells $\overline{R^C}$, as follows: Considering successive antidiagonals, $\big\{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \big| i+j = n+1\big\}$, $n = 1, 2, 3, \ldots$ one at a time, apply the recursive definition:*

$$
\begin{aligned}
\chi^C : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \; & \{0,1\} \\
(i,j) \mapsto \; & \begin{cases} 1, & \sum_{h=1}^{j-1} \chi^C(i,h) < C(i) \text{ and } \sum_{k=1}^{i-1} \chi^C(k,j) < C(j); \\ 0, & \text{otherwise;} \end{cases}
\end{aligned} \tag{15}
$$

where a sum is defined as 0 when its upper limit is less than its lower. Now, suppose C is nondecreasing. Then, the resulting χ^C and R^C are RC convex *(Property 2) and* diagonal containing *(Property 3).*

Proof. Let R^C_n indicate the triangular region formed by cutting the partial construction R_n, per Definition 4.1, by a linear constraint on the n^{th} antidiagonal. That is, let $R^C_n = R^C \cap \big\{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \big| i+j \leq n+1\big\}$, and proceed by induction on n. For $n = 1$, since $C(1)$ takes a positive integer value (Property 1′), the recursion (15) necessarily gives $\chi^C(1,1) = 1$ and $(1,1) \in R^C$.

RC convexity: Suppose R^C_{n-1} is RC convex, but that R^C_n lacks RC convexity. Without loss of generality, suppose that R^C_n lacks row convexity. This would imply that, in some row of R^C_n, blackening started, was interrupted by white cells, and then resumed. That is, $\exists i, j, k$, with $i+j=n+1$ and $k<j-1$, such that $(i,j) \in R_n^C$, $(i,j-1) \notin R_n^C$, and $(i,k) \in R_n^C$. By hypothesis, however, $(i,j-1) \notin R_n^C$ would imply that $\sum_{h=1}^{j-2} \chi^C(i,h) = C(i)$, or that blackening was previously completed in row i, which would contradict the supposed placement of additional black cell $(i,j) \in R_n^C$ in row i by (15).

Diagonal containment: For $n+1$ even, suppose $R_1^C \subset, \dots, \subset R_{n-2}^C$ are diagonal containing, but that R_n^C is missing the diagonal element. That is, suppose $(\frac{n+1}{2}, \frac{n+1}{2}) \notin R_n^C$. Since $C(\frac{n+1}{2})$ takes a positive integer value, this would imply that blackening in row $\frac{n+1}{2}$ and column $\frac{n+1}{2}$ had already been completed in R_{n-1}^C. Thus, following (15), we have $\sum_{h=1}^{\frac{n+1}{2}-1} \chi^C(\frac{n+1}{2}, h) = C(\frac{n+1}{2}) > 0$ and $\sum_{k=1}^{\frac{n+1}{2}-1} \chi^C(k, \frac{n+1}{2}) = C(\frac{n+1}{2}) > 0$, respectively. In particular, for the start of blackening in column $\frac{n+1}{2}$ and row $\frac{n+1}{2}$, let $i = \min\left\{k | (k, \frac{n+1}{2}) \in R_{n-1}^C\right\}$, respectively, $j = \min\left\{h | (\frac{n+1}{2}, h) \in R_{n-1}^C\right\}$, where $i, j \leq \frac{n+1}{2} - 1$.

By hypothesis, R_{n-1}^C contains diagonal elements $(1,1), \dots, (\frac{n+1}{2}-1, \frac{n+1}{2}-1)$, in particular, the diagonal elements (i,i) and (j,j). Suppose $i \leq j$, then by RC convexity, we must have $(i,i), \dots, (i, \frac{n+1}{2}) \in R_{n-1}^C$. However, this would imply that the count of row i exceeds that of row $\frac{n+1}{2} > i$, because $C(\frac{n+1}{2}) = \sum_{h=1}^{\infty} \chi^C(\frac{n+1}{2}, h) = \sum_{h=j}^{\frac{n+1}{2}-1} \chi^C(\frac{n+1}{2}, h) \leqslant \frac{n+1}{2} - j < \frac{n+1}{2} - i + 1 = \sum_{h=i}^{\frac{n+1}{2}} \chi^C(i,h) \leqslant \sum_{h=1}^{\infty} \chi^C(i,h) = C(i)$, which would contradict the assumption that C is nondecreasing.

Likewise, if $j \leq i$, then by RC convexity, we must have $(j,j), \dots, (\frac{n+1}{2}, j) \in R_{n-1}^C$, so that the count of black cells in column j would exceed that of column $\frac{n+1}{2} > j$, because $C(\frac{n+1}{2}) = \sum_{k=1}^{\infty} \chi^C(k, \frac{n+1}{2}) = \sum_{k=i}^{\frac{n+1}{2}-1} \chi^C(k, \frac{n+1}{2}) \leqslant \frac{n+1}{2} - i < \frac{n+1}{2} - j + 1 = \sum_{k=i}^{\frac{n+1}{2}} \chi^C(k,j) \leqslant \sum_{k=1}^{\infty} \chi^C(k,j) = C(j)$. □

By Proposition 6.1, the stair-cone produced by Method 1 (11), in particular, satisfies Properties 2 and 3.

6.2. **Investigation of Method 2.** Firstly, note that Method 2 inherently assures Property 2, because it completes blackening in any row (column) at some finite stage and designates all subsequent cells in that row (column) to remain white. The indexing arrow advances irreversibly northward (eastward) and the completed row (column) never returns as the incumbent. Further, consider in Figure 1 that the boundary of the black stair-cone comprises two *staircases*: A gently-raked staircase on the boundary with W^{lower}, below, and a steeply-raked staircase on the boundary with W^{upper}, above.

As a consequence of the initial pattern $B_2 = \{(1,1),(2,2)\}$ (see Figure 5) and the propagation rule (13), steps of the gentle staircase have a rise of one unit and a going of either one or two units while, by symmetry, steps of the steep staircase have a going of one unit, and a rise of either one or two units. Cells $\{(i, A_i)\}_i = \{(\Omega_j, j)\}_j$ of the upper staircase, and, $\{(A_j, j)\}_j = \{(i, \Omega_i)\}_i$ of the lower, generate the outside corners of B, with (A_j, j) or (i, Ω_i) defining a gentle step and (i, A_i) or (Ω_j, j) defining a steep step.

Secondly, note that the method starts from a symmetric initial pattern and builds on it in a symmetric fashion, applying the same rule (13) to determine row-ends and column-ends. Since every row terminates at the beginning of some column, and, conversely, every column terminates at the beginning of some row, the symmetry of the rule also applies to row-starts and column-starts. Consequently, $\min_{(k,j)\in B} j = A_k = \min_{(i,k)\in B} i$ and $\max_{(k,j)\in B} j = \Omega_k = \max_{(i,k)\in B} i$.

In addition, for steps on the gentle staircase, let $\{(i, M_i)\}_i$ designate cells whose bottom edges form the first unit of the tread. Then for steps on the steep staircase, the left edges of cells $\{(M_j, j)\}_j$ form the first unit of the riser. Another way to conceive M_k is to note that not every column begins at the end of a row, and likewise, not every row begins at the end of a column. Thus, the M_k are the "extra starts" — the starts that are not also ends.

Proposition 6.2. *For $n = 1, 2, \ldots$, Method 2 generates:*

$$\begin{array}{lll} (i) & \Omega_n & = \lfloor n\phi \rfloor, \\ (ii) & M_n & = \lfloor (n-1)\phi \rfloor + 1, \ \textit{and} \\ (iii) & A_n & = \lceil n/\phi \rceil, \end{array}$$

repectively, sequences 000201*,* 026351*, and* 019446 *of* [4].

Proof. (i) The construction method (13) assures the dichotomy $A_{i+1} = A_i$ or $A_{i+1} = A_i + 1$, for all $i \geqslant 1$, *i.e.*, steep steps have a rise of either one or two. Moreover, by symmetry of the method, if the topmost black cell (Ω_j, j) of some column j occurs in row i, this cell must also be the leftmost black cell, (i, A_i), of row i. Conversely, when $A_{i+1} = A_i$, row i clearly cannot contain the topmost black cell of any column. Therefore, $A_{i+1} = A_i + 1$ if and only if row i does contain the topmost black cell of some column, that is, if and only if $i \in \{\Omega_1, \ldots, \Omega_i\}$. Use this symmetry to eliminate A_i from Rule (13) and obtain the univariate recursion

$$\Omega_{i+1} = \begin{cases} \Omega_i + 2, & i \in \{\Omega_1, \ldots, \Omega_i\}; \\ \Omega_i + 1, & i \notin \{\Omega_1, \ldots, \Omega_i\}; \end{cases} i \geqslant 1, \tag{16}$$

together with $\Omega_1 = 1$ a known recursion for the sequence $\Omega_i = \lfloor i\phi \rfloor$ (000201).

(ii) Turning our attention to M_j, observe that each steep step comprises either one or two black cells. Having just established that the upper cell in the riser of steep step $j+1$ is $(\Omega_{j+1}, j+1) = (\lfloor (j+1)\phi \rfloor, j+1)$, now consider the lower cell in the riser of steep step $j+1$. For all $j \geqslant 1$, the lower cell $(M_{j+1}, j+1)$ in the riser of steep step $j+1$ must lie one position above and one position to the right of the previous outside-corner cell, *i.e.*,

$$M_{j+1} = \Omega_j + 1, j \geq 1, \tag{17}$$

giving $(M_{j+1}, j+1)$=$(\Omega_j + 1, j+1)$ for the position of the individual cell, collectively, therefore, $\{M_j\}_{j\geqslant 1} = \{1\} \cup \{\Omega_j + 1\}_{j\geqslant 1}$. Alternatively, substituting (17) into recursion (16), the recursion

$$M_{j+1} = \begin{cases} 1, & j = 0; \\ 2, & j = 1; \\ M_j + 2, & j \in \{M_2, \ldots, M_j\}, \quad j \geq 2; \\ M_j + 1, & j \notin \{M_2, \ldots, M_j\}, \quad j \geq 2; \end{cases} \tag{18}$$

obtains. This matches the recursion for $M_j = \lfloor (j-1)\phi \rfloor + 1$ (026351).

(iii) Considering sequence (A_j) as column starts, an integer that repeats (exactly twice) corresponds to a long step — a step with a going of two on the gentle staircase. For a long step in row $i+1$, we express this repetition by $\Omega_{i+1} = M_{i+1}+1$. Substituting (17) gives the condition $\Omega_{i+1} = \Omega_i+2$ on $i+1$ equivalent by (16) to the condition $i \in \{\Omega_1, \ldots, \Omega_i\}$ on i, for $i \geqslant 1$. Hence $\{\Omega_i + 1\}$ contains those numbers appearing twice in sequence (A_j).

Conversely, an integer that appears only once in the sequence (A_j) of column starts corresponds to a narrow step — a step with a going of one on the gentle staircase. For narrow step $i+1$, we express this uniqueness by $\Omega_{i+1} = M_{i+1}$. Substituting into (17) gives the condition $\Omega_{i+1} = \Omega_i + 1$ on $i+1$ equivalent by (16) to the condition $i \notin \{\Omega_1, \ldots, \Omega_i\}$ on i, for $i \geq 1$. Thus, $\overline{\{\Omega_i\}} \equiv \mathbb{Z}_+ \setminus \{\Omega_i\}$ gives integers appearing exactly once in sequence A_j, corresponding to narrow steps $i+1$ for $i+1 \geqslant 2$.

In addition, $\Omega_1 = M_1 = A_1 = 1$ for the first narrow step. Therefore $\{1\} \cup \overline{\{\Omega_i\}}$ completely describes elements that appear exactly once in sequence A_j. Identifying its unique and repeated elements with known formulae shows A_j equivalent to Sloane's 019446, given by $A_j = \lceil j/\phi \rceil$. □

Remark 6.1. The weak method of constructing the stair-cone provides a graphical demonstration of self-similarity / recursion properties for series given in Sloane [4]. Specifically, it demonstrates (16) for 000201, (18) for 026351, and for 019446, that A_i comprises two copies each of $\{\Omega_i + 1\}$ and one each of $\{1\} \cup \overline{\{\Omega_i\}}$.

Lemma 6.3. *(A Figure with starts A_k and ends Ω_k is symmetric) Having $A_k = \lceil k/\phi \rceil$, respectively, $\Omega_k = \lfloor k\phi \rfloor$ as the starting, respectively, ending cells of rows (columns) implies that the same is also true for columns (rows), that is, $\{(i,j) \mid \lceil i/\phi \rceil \leq j \leq \lfloor i\phi \rfloor\} = \{(i,j) \mid \lceil j/\phi \rceil \leq i \leq \lfloor j\phi \rfloor\}$. In particular, the resulting figure is symmetric: $(i,j) \in B \Leftrightarrow (j,i) \in B$.*

Proof. Symmetry follows from the fact that $\forall k \in \mathbb{Z}$, $k = \lceil \lfloor k\phi \rfloor / \phi \rceil$. □

Proposition 6.4. *(Converse of Proposition 4.2) $A_k = \lceil k/\phi \rceil$ and $\Omega_k = \lfloor k\phi \rfloor$ for a stair tiling imply Properties 1, 2, and 3.*

Proof. The known identity $n = \lfloor n\phi \rfloor - \lceil n/\phi \rceil + 1$, together with Lemma 6.3, show that Property 1 is satisfied both row-wise and column-wise. Property 2 is trivially satisfied. Finally, Property 3 follows from $\lceil n/\phi \rceil < n < \lfloor n\phi \rfloor$. □

6.3. **Investigation of Method 3.** Begin the investigation of Method 3 by observing how the quilt evolves with each action u of the method. Investigate the black region and track progress of the method by maintaining a ledger of black squares located on or below the spine. Initially, the ledger only contains entries for the spinal squares themselves, which, by convention, arise through action $u = 0$. Subsequent actions $u \geq 1$ add squares to the quilt, and with each square added, a new entry in the ledger. Sort the ledger with entries for the smallest squares — those consisting of one cell — first, followed entries for the second-smallest square (size 2×2), and so forth, through the k^{th}-smallest squares, of size $F_{k+1} \times F_{k+1}$. Each entry comprises the size of the square, the interval it covers, $[a_{k,n}, b_{k,n}] \times [c_{k,n}, d_{k,n}]$, and finally its genealogy, $v_{n,k}$. (Recall that the *genealogy* is an integer tuple recording the generations u at which a square and each of its ancestors first appeared.) Figure 8 showed action $u = 2$ of Method 3, following which, the top of the ledger would look like Table 5.

Recall from Section 5.3 that action u of Method 3 acts on spinal square $S_{0,u}$ together with all other black squares $S_{n,k}$ that lie (i) east of spinal square $S_{0,u-1}$ and (ii) south of spinal square $S_{0,u+1}$. That is, all candidate squares $S_{n,k}$ satisfy,

Ordinal Size, Dimensions k^{th}-smallest, or $F_{k+1} \times F_{k+1}$	Rows $\times$ Columns $[a_{n,k}, b_{n,k}] \times [c_{n,k}, d_{n,k}]$	Genealogy $v_{n,k}$
Smallest, or 1×1	$S_{0,1} = [1,1] \times [1,1]$	$v_{0,1} = ()$
Smallest, or 1×1	$S_{1,1} = [3,3] \times [4,4]$	$v_{1,1} = (1)$
Smallest, or 1×1	$S_{2,1} = [6,6] \times [9,9]$	$v_{2,1} = (1,2)$
2^{nd}-smallest, or 2×2	$S_{0,2} = [2,3] \times [2,3]$	$v_{0,2} = ()$
2^{nd}-smallest, or 2×2	$S_{1,2} = [5,6] \times [7,8]$	$v_{1,2} = (2)$
3^{rd}-smallest, or 3×3	$S_{0,3} = [4,6] \times [4,6]$	$v_{0,3} = ()$

TABLE 5. Partial Ledger of black quilt squares after action $u = 2$ of Method 3

respectively (i) $c_{n,k} \geq d_{0,u-1} + 1 = c_{0,u}$ and (ii) $b_{n,k} \leq a_{0,u+1} - 1 = b_{0,u}$. As spinal squares increase in size along the spine, their northern indices $b_{0,u}$ lie farther above the diagonal. Every square in the quilt descends from some spinal-square ancestor. Successive actions u shift the square by $F_{u+2} \times F_{u+3}$ — a shift of more columns than rows, that is, always farther east than north. Applying such shifts to a spinal square can never cause it to land north of a subsequent spinal square. Thus, for the current action $u \geq k$, if candidate square $S_{n,k}$ already appears in the ledger, then $b_{n,k} \leq b_{0,u}$, and candidate $S_{n,k}$ automatically satisfies the "southness" that Method 3 requires. So, it suffices to test the "eastness" of a candidate square for reproduction.

Consequently, each action $u \geq 1$ of Method 3 on the quilt corresponds to the following update of the ledger: First, test each entry $S_{n,k}$ currently in the ledger for "eastness," then clone the entries that pass the test, as follows. If the western index $c_{n,k}$ of $S_{n,k}$ satisfies

$$c_{n,k} \geq c_{0,u} = F_{u+2} - 1, \tag{19}$$

then insert a modified clone of the entry in the ledger. Modify the cloned entry from its source by (i) a northward shift of F_{u+2}, (ii) an eastward shift of F_{u+3}, and (iii) by appending index u to the genealogy.

Now, consider the ledger after a large number of actions $u \gg k$. These actions reproduce the spinal square ancestor $S_{0,k}$, having $v_{0,k} = ()$. For its descendants $S_{n,k}$, $n = 1, 2, 3 \ldots$, consider the corresponding ledger entries (Table 6), which would immediately follow the entry for $S_{0,k}$ in the ledger:
Table 6 grouped each generation u within a brace. Now, write each generation compactly as

$$S_{n,k} = \begin{cases} S_{n-1,k} + F_{k+2} \times F_{k+3}, & n = 1, \\ S_{n-1,k} + F_{k+3} \times F_{k+4}, & n = 2, \\ S_{n-2,k} + F_{k+4} \times F_{k+5}, & 3 \leq n < 5, \\ S_{n-3,k} + F_{k+5} \times F_{k+6}, & 5 \leq n < 8, \\ \vdots & \vdots \end{cases}$$

or even more compactly, write all generations as

$$S_{n,k} = S_{n-F_{u-k+1},k} + F_{u+2} \times F_{u+3}, F_{u-k+2} \leq n < F_{u-k+3}, u = k, k+1, \ldots,$$

and further simplify, by the substitution $t = u - k + 1$, to yield the following result:

Interval	Genealogy	
$S_{1,k} = S_{0,k} + F_{k+2} \times F_{k+3}$	$v_{1,k} = (k)$	$u = k,$
$S_{2,k} = S_{1,k} + F_{k+3} \times F_{k+4}$	$v_{2,k} = (k, k+1)$	$u = k+1,$
$S_{3,k} = S_{1,k} + F_{k+4} \times F_{k+5}$	$v_{3,k} = (k, k+2)$	$u = k+2,$
$S_{4,k} = S_{2,k} + F_{k+4} \times F_{k+5}$	$v_{4,k} = (k, k+1, k+2)$	
$S_{5,k} = S_{2,k} + F_{k+5} \times F_{k+6}$	$v_{5,k} = (k, k+1, k+3)$	$u = k+3,$
$S_{6,k} = S_{3,k} + F_{k+5} \times F_{k+6}$	$v_{6,k} = (k, k+2, k+3)$	
$S_{7,k} = S_{4,k} + F_{k+5} \times F_{k+6}$	$v_{7,k} = (k, k+1, k+2, k+3)$	
$\vdots$	$\vdots$	

Table 6. Ledger of black squares of size $F_{k+1} \times F_{k+1}$ produced by quilting action u

Lemma 6.5 (Recurrence for constructing the black region of the quilt)**.** *The quilt comprises subsequences of squares sized $F_{k+1} \times F_{k+1}$, for $k = 1, 2, 3, \ldots$, described by the recurrence:*

$$
\begin{aligned}
S_{n,k} =& S_{n-F_t,k} + F_{t+k+1} \times F_{t+k+2}, F_{t+1} \le n < F_{t+2}, t = 1, 2, \ldots; \ \textit{where} \\
S_{0,k} =& [a_{0,k}, b_{0,k}] \times [c_{0,k}, d_{0,k}] = [F_{k+2} - 1, F_{k+3} - 2] \times [F_{k+2} - 1, F_{k+3} - 2].
\end{aligned} \tag{20}
$$

Note that t in Lemma 6.5 has the following interpretation. Because spinal squares increase in size and touch diagonally, each lying strictly northward and eastward of the previous one, action $u = k$ is the first action to involve the k^{th} square on the spine. Thus, no prior action $u < k$ of the method could have involved a square of ordinal size k. Therefore, for a given value of k, t counts the actions that reproduce squares $S_{n,k}$ of ordinal size k.

Further, by the substitution $t = u - k + 1$, test (19) becomes:

$$c_{n,k} \ge F_{t+k+1} - 1. \tag{21}$$

Definition 6.1 (Cohorts of quilt squares)**.** Define the *cohort* $C_{t,k}$ of quilt squares as the sequence of quilt squares of ordinal size k placed by action t. Per (20), the method begins with the spine, defining $C_{0,k} \equiv (S_{0,k})$, and subsequently, $C_{t,k} = (S_{F_{t+1},k}, \ldots, S_{F_{t+2}-1,k})$.

Whereas the generation index u provides an absolute count of stages in Method 3, the cohort index t counts generations relative to a family $(S_{n,k})_{n=0}^{\infty}$ or $(R_{n,k})_{n=0}^{\infty}$, for a specific value of k.

In view of Definition 6.1, the condition $F_{t+1} \le n < F_{t+2}$ on recurrence (20) limits the reproductive domain of action t to squares of cohorts $C_{t-2,k}$ and $C_{t-1,k}$, which, by recurrence (20) itself, produces the squares of cohort $C_{t,k}$. It remains to show that all the candidate squares $S_{n,k} \in C_{t-2,k} \cup C_{t-1,k}$ in the two previous cohorts satisfy test (21) and any other existing candidate squares $S_{n,k} \in C_{0,k} \cup \cdots \cup C_{t-3,k}$ in earlier cohorts fail the test. This will be the subject of the next proposition.

Proposition 6.6 will confirm that the recurrence (20) is injective, in particular, that each quilt square $S_{n,k}$, lies strictly north and strictly east of the previous one $S_{n-1,k}$. Thus, the proposition will show that the recurrence (20) is sufficient to describe Method 3 of constructing the quilt, making the test (21) redundant.

Proposition 6.6. *For integer $k \ge 1$, consider the k^{th}-smallest sized squares in the quilt, those comprising $F_{k+1} \times F_{k+1}$ cells. Let $n = 0, 1, 2, \ldots$, and index the squares*

$S_{n,k} \equiv [a_{n,k}, b_{n,k}] \times [c_{n,k}, d_{n,k}]$. *Then the conditions on the recurrence (20) describe the all candidates that satisfy test (21) and only candidates that satisfy test (21). Moreover, the resulting sequence* $(S_{n,k})_{n=0}^{\infty}$ *strictly increases, "spreading out" the quilt squares and making recurrence (20) injective.*

Proof. For arbitrary cohort $C_{t,k}$ of quilt squares $(S_{F_{t+1},k}, \ldots, S_{F_{t+2}-1,k})$, the proof will show the following:

(22a) $c_{m-F_t,k} < F_{t+k+1} - 1 \qquad m < F_{t+1}$

(22b) $c_{n-F_t,k} \geq F_{t+k+1} - 1 \qquad F_{t+1} \leq n < F_{t+2}$

(22c) $a_{n,k} = a_{n-F_t,k} + F_{t+k+1} \qquad F_{t+1} \leq n < F_{t+2}$

(22d) $c_{n,k} = c_{n-F_t,k} + F_{t+k+2} \qquad F_{t+1} \leq n < F_{t+2}$

(22e) $a_{n,k} > a_{m,k} + F_{k+1} - 1 = b_{m,k} \qquad m < F_{t+1} \leq n < F_{t+2}$

(22f) $c_{n,k} > c_{m,k} + F_{k+1} - 1 = d_{m,k} \qquad m < F_{t+1} \leq n < F_{t+2}$

Here, (22a), respectively, (22b), show that condition $F_{t+1} \leq n < F_{t+2}$ on the recurrence (20) is sufficient, respectively, necessary to describe all candidate squares that satisfy the test (21). The components (22c) and (22d) of (20) thus become necessary and sufficient to describe the squares of cohort $C_{t,k}$. Finally, (22e) and (22f) show that the quilt squares "spread out" in two dimensions, making the recurrence injective.

Base case: First consider the zeroth cohort $C_{0,k}$. By Definition 6.1, it is a singleton comprising $S_{0,k} = [a_{0,k}, b_{0,k}] \times [c_{0,k}, d_{0,k}] = [F_{k+2} - 1, F_{k+3} - 2] \times [F_{k+2} - 1, F_{k+3} - 2]$. Then, for $t = 1$,

(23a) $\nexists S_{m,k}, m < 0,$

(23b) $c_{0,k} = F_{k+2} - 1 \geq F_{k+2} - 1,$

(23c) $a_{1,k} = a_{0,k} + F_{k+2},$

(23d) $c_{1,k} = c_{0,k} + F_{k+3},$

(23e) $a_{1,k} = 2F_{k+2} - 1 > F_{k+3} - 2 = a_{0,k} + F_{k+1} - 1 = b_{0,k},$

(23f) $c_{1,k} = F_{k+4} - 1 > F_{k+3} - 2 = c_{0,k} + F_{k+1} - 1 = d_{0,k},$

where (23b) shows that, by definition, $S_{0,k}$ satisfies test (21), and (23a) shows the absence of further candidates for reproduction. This permits $S_{0,k}$ to reproduce via (20), producing in $S_{1,k}$, in particular, its extrema (23c) and (23d). These, in turn satisfy (22e) and (22f), showing that $S_{1,k}$ has "spread out" from $S_{0,k}$.

Thus, the first cohort $C_{1,k}$ is also a singleton comprising $S_{1,k} = S_{0,k} + F_{k+2} \times F_{k+3} = [2F_{k+2} - 1, F_{k+4} - 2] \times [F_{k+4} - 1, 2F_{k+3} - 2] \equiv [a_{1,k}, b_{1,k}] \times [c_{1,k}, d_{1,k}]$. For $t = 2$,

(24a) $c_{0,k} = F_{k+2} - 1 < F_{k+3} - 1$

(24b) $c_{1,k} = F_{k+4} - 1 \geq F_{k+3} - 1$

(24c) $a_{2,k} = a_{1,k} + F_{k+3}$

(24d) $c_{2,k} = c_{1,k} + F_{k+4}$

(24e) $a_{2,k} = F_{k+4} + F_{k+2} - 1 > F_{k+4} - 2 = a_{1,k} + F_{k+1} - 1 = b_{1,k}$

(24f) $c_{2,k} = 2F_{k+4} - 1 > 2F_{k+3} - 2 = c_{1,k} + F_{k+1} - 1 = d_{1,k}$

where (24b) shows that, by definition, $S_{0,k}$ fails test (21) for $t = 2$, whereas $S_{1,k}$ passes, showing that the conditions on (20) are sufficient, and allowing (only) $S_{1,k}$ to reproduce. The resultant offspring, $S_{2,k}$, has extrema (24c) and (24d), and is spaced away from $S_{1,k}$ ((24e) and (24f)). Thus, the second cohort $C_{2,k}$ is also a singleton comprising $S_{2,k} = S_{1,k} + F_{k+2} \times F_{k+3} = [F_{k+4} + F_{k+2} - 1, F_{k+5} - 2] \times [2F_{k+4} - 1, F_{k+5} + F_{k+3} - 2] = [a_{2,k}, b_{2,k}] \times [c_{2,k}, d_{2,k}]$.

The proposition thus being true for the first several relevant actions, t, of the method, now consider induction on t. Suppose that the claims are true for cohorts $t-3$, $t-2$, and $t-1$. That is, respectively,

(25a) $c_{m-F_{t-3},k} < F_{t+k-2} - 1 \qquad m < F_{t-2}$

(25b) $c_{n-F_{t-3},k} \geq F_{t+k-2} - 1 \qquad F_{t-2} \leq n < F_{t-1}$

(25c) $a_{n,k} = a_{n-F_{t-3},k} + F_{t+k-2} \qquad F_{t-2} \leq n < F_{t-1}$

(25d) $c_{n,k} = c_{n-F_{t-3},k} + F_{t+k-1} \qquad F_{t-2} \leq n < F_{t-1}$

(25e) $a_{n,k} > a_{m,k} + F_{k+1} - 1 \qquad m < F_{t-2} \leq n < F_{t-1}$

(25f) $c_{n,k} > c_{m,k} + F_{k+1} - 1 \qquad m < F_{t-2} \leq n < F_{t-1}$

(26a) $c_{m-F_{t-2},k} < F_{t+k-1} - 1 \qquad m < F_{t-1}$

(26b) $c_{n-F_{t-2},k} \geq F_{t+k-1} - 1 \qquad F_{t-1} \leq n < F_t$

(26c) $a_{n,k} = a_{n-F_{t-2},k} + F_{t+k-1} \qquad F_{t-1} \leq n < F_t$

(26d) $c_{n,k} = c_{n-F_{t-2},k} + F_{t+k} \qquad F_{t-1} \leq n < F_t$

(26e) $a_{n,k} > a_{m,k} + F_{k+1} - 1 \qquad m < F_{t-1} \leq n < F_t$

(26f) $c_{n,k} > c_{m,k} + F_{k+1} - 1 \qquad m < F_{t-1} \leq n < F_t$

(27a) $c_{m-F_{t-1},k} < F_{t+k} - 1 \qquad m < F_t$

(27b) $c_{n-F_{t-1},k} \geq F_{t+k} - 1 \qquad F_t \leq n < F_{t+1}$

(27c) $a_{n,k} = a_{n-F_{t-1},k} + F_{t+k} \qquad F_t \leq n < F_{t+1}$

(27d) $c_{n,k} = c_{n-F_{t-1},k} + F_{t+k+1} \qquad F_t \leq n < F_{t+1}$

(27e) $a_{n,k} > a_{m,k} + F_{k+1} - 1 \qquad m < F_t \leq n < F_{t+1}$

(27f) $c_{n,k} > c_{m,k} + F_{k+1} - 1 \qquad m < F_t \leq n < F_{t+1}$

Now show the same relationships (22a) through (22f) for cohort t.

(a) It suffices to show (22a) for $m = F_{t+1} - 1$, or $c_{F_{t-1}-1,k} < F_{t+k+1} - 1$. To show the latter, take $m = F_t - 1$ in inequality (27a) and $n = F_{t-1} - 1$ in formula (25d) and combine these to obtain $c_{F_{t-1}-1,k} < F_{t+k} + F_{t+k-2} - 1 < F_{t+k+1} - 1$.

(b) For (22b), show separately that $c_{m-F_t,k} \geq F_{t+k+1} - 1, F_{t+1} \leq m < 2F_t$ and that $c_{m-F_t,k} \geq F_{t+k+2} - 1, 2F_t \leq m < F_{t+2}$. For the former, substitute formula (26d) into inequality (26b) and let $n = m - F_t$. For the latter, substitute formula (27d) into inequality (27b) and let $n = m - F_t$.

(c) and (d) With the condition $F_{t+1} \leq n < F_{t+2}$ being necessary (22b) and sufficient (22a), the recurrences (22c) and (22d) become valid for producing cohort t.

(e) It suffices to show (22e) for $m = F_{t+1} - 1, n = F_{t+1}$. To obtain $a_{F_{t+1},k} > a_{F_{t+1}-1,k} + F_{k+1} - 1$, first take $m = F_{t-1} - 1, n = F_{t-1}$ in inequality (26e) to obtain

$$a_{F_{t-1},k} > a_{F_{t-1}-1,k} + F_{k+1} - 1. \tag{28}$$

Take $n = F_{t+1}$ in (22c) to obtain $a_{F_{t-1}} = a_{F_{t+1}} - F_{t+k+1}$ and substitute into the left-hand side of (28). Next, take $n = F_t - 1$ in (26c) to obtain $a_{F_{t-1}-1} = a_{F_t-1} - F_{t+k-1}$ and substitute into the right-hand side of (28). Finally, take $n = F_{t+1} - 1$ in (27c) to obtain $a_{F_t-1} = a_{F_{t+1}-1} - F_{t+k}$ and substitute into the right-hand side again to yield the desired inequality.

(f) It suffices to show (22f) for $m = F_{t+1} - 1, n = F_{t+1}$. To obtain $c_{F_{t+1},k} > c_{F_{t+1}-1,k} + F_{k+1} - 1$, first take $m = F_{t-1} - 1, n = F_{t-1}$ in inequality (26f) to obtain

$$c_{F_{t-1},k} > c_{F_{t-1}-1,k} + F_{k+1} - 1. \tag{29}$$

Take $n = F_{t+1}$ in formula (22d) to obtain $c_{F_{t-1}} = c_{F_{t+1}} - F_{t+k+2}$ and substitute into the left-hand side of (29). Next, take $n = F_t - 1$ in (26d) to obtain $c_{F_{t-1}-1} = c_{F_t-1} - F_{t+k}$ and substitute into the right-hand side of (29). Finally, take $n = F_{t+1} - 1$ in formula (27d) to obtain $c_{F_t-1} = c_{F_{t+1}-1} - F_{t+k+1}$ and substitute into the right-hand side again to yield the desired inequality. □

Remark 6.2. Like the intervals of quilt squares shown in Table 6, their genealogies can be written compactly, by cohort:

$$v_{n,k} = \begin{cases} v_{n-1,k} \oplus (k), & n = 1, \\ v_{n-1,k} \oplus (k+1), & n = 2, \\ v_{n-2,k} \oplus (k+2), & 3 \le n < 5, \\ v_{n-3,k} \oplus (k+3), & 5 \le n < 8, \\ \vdots & \vdots \end{cases}$$

or even more compactly as:

$$v_{n,k} = v_{n-F_{u-k+1},k} \oplus (u), F_{u-k+2} \le n < F_{u-k+3}, u = k, k+1, \ldots,$$

and further simplified, by the substitution $t = u - k + 1$, to

$$v_{n,k} = v_{n-F_t,k} \oplus (t + k - 1), F_{t+1} \le n < F_{t+2}, t = 1, 2, \ldots,$$

where $v_{0,k} = ()$, analogous to (20).

Remark 6.3. Similar to (20), one can show for the rectangles of the white region that:

$$\begin{aligned} R_{n,k} =& R_{n-F_t,k} + F_{t+k} \times F_{t+k+1}, F_{t+1} \le n < F_{t+2}, t = 2, 3, \ldots; \text{ where} \\ R_{1,k} =& [\alpha_{1,k}, \beta_{1,k}] \times [\gamma_{1,k}, \delta_{1,k}] = [1, F_{k+1}] \times [F_{k+3} - 1, F_{k+4} - 2]. \end{aligned}$$

Corollary 6.7. *As in Definition 6.1, let $C_{u,k}$ be the set of squares having edge length F_{k+1} deposited on the quilt by action u of Method 3. Then action u deposits $F_u, \ldots, F_1$ squares of edge lengths $F_2, \ldots, F_{u+1}$, respectively, or*

$$|C_{u,k}| = \begin{cases} 0, & u < k \\ F_{u-k+1}, & u \ge k \end{cases}$$

Substituting $t = u - k + 1$ gives, for cohort $C_{t,k}$,

$$|C_{t,k}| = F_t, t \ge 1.$$

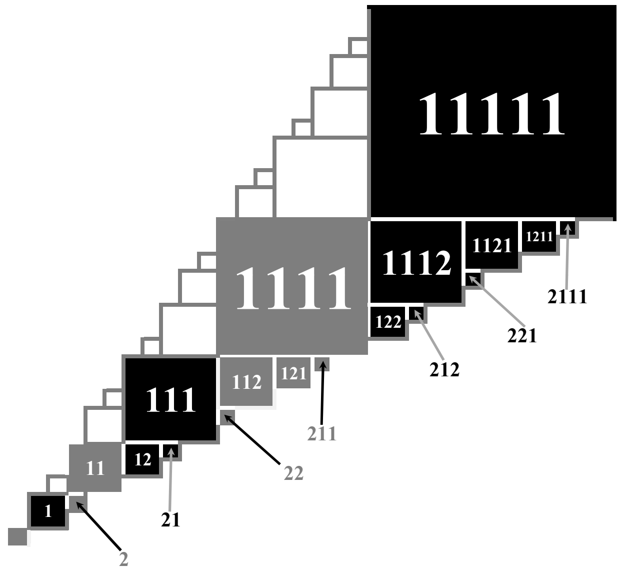

FIGURE 10. Restricted integer compositions from genealogies of quilt squares: Referring to Method 3, the first clone of a square receives a suffix of 1 while the second clone of a square receives a suffix of 2. The quilt squares thus enumerate all integer compositions that use only 1's and 2's [2].

Similarly, for the white region, action u deposits $F_{u+1}, \ldots, F_1$ rectangles of dimensions $F_2 \times F_3, \ldots, F_{k+1} \times F_{k+2}$, or, letting $D_{u,k}$ be the set of quilt rectangles having dimensions $F_{k+1} \times F_{k+2}$, deposited by quilt action u of Method 3.

$$|D_{u,k}| = \begin{cases} 0, & u < k; \\ F_{u-k+2}, & u \geq k. \end{cases}$$

Substituting $t = u - k + 1$ gives, for cohort $D_{t,k}$,

$$|D_{t,k}| = F_{t+1}, t \geq 1.$$

7. Conclusions

In this first of three parts, the paper examined a stair-cone tiling (Figure 1) and a quilt tiling (Figure 2), providing multiple methods for constructing the tilings and demonstrating equivalence for two different constructions of the stair-cone. Part 3 of the paper [3] will complete the formal proof that the stair-cone and quilt are equivalent (Corollary 3.5), drawing on developments made in part 2 [2].

The paper focused on a specific, canonical "Fibonacci stair-cone," which exhibits numerous instances of the Fibonacci sequence. However, the discussion also described a non-canonical Fibonacci stair-cone (Example 4.1), and a non-Fibonacci stair-cone (a stair-cone after Pell, Figure 4).

For the "quilt after Fibonacci," the paper described the integer sequences that specify the extrema of its square and rectangular patches, as well as their construction order. Thus, the quilt also provided a graphical calculus for identities between integer sequences — a role it continues to play in parts 2 and 3 of the paper.

Parts 2 and 3 further study the quilt, especially the sequences $a_{n,k}$, $b_{n,k}$, $c_{n,k}$, $d_{n,k}$, $\alpha_{n,k}$, $\beta_{n,k}$, $\delta_{n,k}$ and $\gamma_{n,k}$. Part 2 focuses on cohorts of these quilt sequences, and more general cohort sequences, including cohorts of tuples and of functions, in the context of iterated floor functions, numeration systems, and restricted composition of integers such as those illustrated by Figure 10.

Part 3 focuses on the quilt sequences as two-dimensional arrays of numbers (Tables 1, 2, 3, and 4, here), showing these arrays to be either interspersion–dispersion arrays, or to satisfy a relaxed definition of *interspersoid–dispersoid* array. In both cases, the paper characterizes the structure of blocks of rows with respect to interspersion, demonstrating yet one more element of self-similarity arising from the quilt. Part 3 will also use the quilt as a means of visualizing complementary equations between iterated floor functions of the Wythoff type (those involving ϕ).

References

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